

# Generating sequences and Poincaré series for a finite set of plane divisorial valuations \*

F. Delgado      C. Galindo      A. Núñez

## Abstract

Let  $V$  be a finite set of divisorial valuations centered at a 2-dimensional regular local ring  $R$ . In this paper we study its structure by means of the semigroup of values,  $S_V$ , and the multi-index graded algebra defined by  $V$ ,  $\text{gr}_V R$ . We prove that  $S_V$  is finitely generated and we compute its minimal set of generators following the study of reduced curve singularities. Moreover, we prove a unique decomposition theorem for the elements of the semigroup. The comparison between valuations in  $V$ , the approximation of a reduced plane curve singularity  $C$  by families of sets  $V^{(k)}$  of divisorial valuations, and the relationship between the value semigroup of  $C$  and the semigroups of the sets  $V^{(k)}$ , allow us to obtain the (finite) minimal generating sequences for  $C$  as well as for  $V$ .

We also analyze the structure of the homogeneous components of  $\text{gr}_V R$ . The study of their dimensions allows us to relate the Poincaré series for  $V$  and for a general curve  $C$  of  $V$ . Since the last series coincides with the Alexander polynomial of the singularity, we can deduce a formula of A'Campo type for the Poincaré series of  $V$ . Moreover, the Poincaré series of  $C$  could be seen as the limit of the series of  $V^{(k)}$ ,  $k \geq 0$ .

---

\*Math. Subject Class. 14B05, 16W50, 16W70, 13A18

Supported by the Spain Ministry of Education MTM2004-00958 and JCyL VA025A07. Second author also supported by Bancaixa P1-1A2005-08.

The authors would like to thank the referee for his/her detailed and helpful comments.

### Addresses:

F. Delgado & A. Núñez: Dto. de Álgebra, Geometría y Topología. Universidad de Valladolid. 47005 Valladolid. Spain. e-mail: fdelgado@agt.uva.es, anunez@agt.uva.es  
C. Galindo: Dto. de Matemáticas (ESTCE), UJI, Campus Riu Sec. 12071 Castellón. Spain. e-mail: galindo@mat.uji.es

# Introduction

This paper deals with the structure of a finite number of divisorial valuations centered at a regular local ring of dimension two. In singularity theory there are many problems that involve finitely many interrelated exceptional divisors (and so, their corresponding divisorial valuations), which cannot be analyzed independently without losing some information. Classification of sandwiched singularities, minimal resolutions and the Nash problem are examples of this situation. The study of plane curve singularities constitutes a similar situation and the treatment of a branch is rather different of the one of the whole curve (see [5] and [6]). Problems as uniformization and monomialization of valuations, studied historically by Zariski and Abhyankar, are also object of recent activity (see e.g. [19]), providing another motivation to our study.

This paper is inspired in two sources. Firstly, the results for the case of a single divisorial valuation by Spivakovsky [18], where minimal generating sequences were computed (see also [15]), and Galindo, who computes the Poincaré series [14]. The second one is the set of results [4, 5, 6] obtained by Campillo, Delgado and Gusein-Zade for a plane curve singularity with several branches where generation of the semigroup, zeta function and Poincaré polynomial are considered.

Throughout this paper, we will assume that  $(R, \mathfrak{m})$  is a local, regular, complete and 2-dimensional ring and that it has an algebraically closed coefficient field. Replacing curves defined by elements in  $R$  by analytically reduced curves defined by elements in the completion  $\hat{R}$  of  $R$ , and considering the valuations defined by their branches in the ring  $R$  (see e.g. [10]), theorems stated in the paper remain true without the assumption of completeness. However, we will consider the complete case because it simplifies the proofs and gives a more clear intuition.

Each irreducible component  $E_\alpha$  of the exceptional divisor  $E$  of a modification  $\pi$  of  $\operatorname{Spec} R$  defines a valuation of the fraction field of  $R$  centered at  $R$ , named divisorial and denoted by  $\nu_\alpha$ . An irreducible element in  $R$  such that the strict transform by  $\pi$  of the corresponding curve is smooth and intersect transversely  $E_\alpha$  at a smooth point of  $E$  is, generically, denoted by  $Q_\alpha$  and plays an important role in this paper. Consider a finite set  $V = \{\nu_1, \dots, \nu_r\}$  of  $r \geq 1$  divisorial valuations associated to exceptional components  $E_{\alpha(i)}$  of  $E$  where  $\pi : (X, E) \rightarrow (\operatorname{Spec} R, \mathfrak{m})$  is the minimal modification such that  $E_{\alpha(i)} \subset E$  for  $1 \leq i \leq r$ . We define the semigroup of values of  $V$  as the sub-semigroup of  $\mathbb{Z}_{\geq 0}^r$  given by  $S_V := \{\underline{\nu}(f) := (\nu_1(f), \dots, \nu_r(f)) \mid f \in R \setminus \{0\}\}$ . A general curve of  $V$  is a reduced plane curve with  $r$  branches each one defined by an equation  $Q_{\alpha(i)} = 0$ . When  $r = 1$ ,  $S_V$  coincides with the

semigroup of values of any general curve of  $V$  [20]. The valuation ideals  $J(\underline{m}) = \{g \in R \mid \underline{\nu}(g) \geq \underline{m}\}$  define a multi-index filtration of the ring  $R$  which gives rise to a graded algebra  $\text{gr}_V R = \bigoplus_{\underline{m} \in \mathbb{Z}_{\geq 0}^r} J(\underline{m})/J(\underline{m} + \underline{e})$ ,

$\underline{e} = (1, \dots, 1)$ . This paper analyzes both objects, the semigroup and the graded algebra, for a set valuations  $V$  looking for its essential arithmetical and algebraic properties.

A description of the semigroup  $S_V$  is given in Section 2. In Theorem 1, we give the minimal set of generators of  $S_V$ , proving that it is finitely generated, unlike the case of a reduced plane curve singularity (see [8] and [4]). The set  $\{B^i := \underline{\nu}(Q_{\alpha(i)}) \mid i = 1, \dots, r\}$  plays a very special role in  $S_V$ . Proposition 6 shows that the projectivization of the vector space  $D(B^i) = J(B^i)/J(B^i + \underline{e})$  is canonically isomorphic to the exceptional divisor  $E_{\alpha(i)}$  which defines the valuation  $\nu_i$ . In particular,  $D(B^i)$  is bidimensional. The study of the dimension  $d_i(\underline{m})$  of the spaces  $D_i(\underline{m}) = J(\underline{m})/J(\underline{m} + e_i)$  allows to prove Theorem 3, which gives a unique decomposition for the elements in  $S_V$  in terms of the set  $\{B^1, \dots, B^r\}$ . In particular, we give another proof of the fact that if  $V$  consists of all the divisors of a modification, then  $S_V$  is a free semigroup generated by  $B^1, \dots, B^r$ .

In Section 3 we describe a generating sequence for a finite set  $V$  of divisorial valuations and for a reduced curve with several branches. Denote by  $\mathcal{E}$  the set of end divisors of the minimal resolution of  $V$ , i.e., the exceptional components  $E_\alpha$  such that  $E \setminus E_\alpha$  is connected, and set  $\Lambda_{\mathcal{E}} = \{Q_\rho \mid E_\rho \in \mathcal{E}\}$ . The main result of this section, Theorem 5, states that  $\Lambda_{\mathcal{E}}$  is a minimal generating sequence of  $V$ , that is, any valuation ideal is generated by monomials in the set  $\Lambda_{\mathcal{E}}$ . After a result of Campillo and Galindo [7], this is equivalent to the fact that  $\text{gr}_V R$  is the  $R/\mathfrak{m}$ -algebra generated by the classes of the elements in  $\Lambda_{\mathcal{E}}$ .

A similar result is true for the set  $W$  of valuations defined by the branches of a reduced plane curve  $C$ , however we must change the set  $\Lambda_{\mathcal{E}}$  by another one  $\Lambda_{\overline{\mathcal{E}}}$  which is also finite (see, again, Theorem 5). The key to understand it, is that  $W$  can be regarded as a limit of families of divisorial valuations  $V^{(k)}$ ,  $k \geq 0$ , and so  $\Lambda_{\overline{\mathcal{E}}}$  as the limit of the sequence  $\Lambda_{\mathcal{E}^{(k)}}$  given by  $V^{(k)}$ . The number of classes in  $\text{gr}_V R$  produced by each element in  $\Lambda_{\mathcal{E}}$  is finite, however some elements in  $\Lambda_{\overline{\mathcal{E}}}$  give infinitely many different classes in the corresponding algebra (see the last remark of Section 3). This fact explains the apparent contradiction between the infinite generation of the semigroup of a plane curve singularity and the existence of a finite generating sequence.

It is worthwhile to mention that the so called multipliers ideals of ideals in the ring  $R$ , can be regarded as ideals  $J(\underline{m})$  for concrete sets,  $V$ , and

elements  $\underline{m}$  [16]. Notice that, in our case, these ideals are exactly the complete ones [17].

The dimensions  $d(\underline{m}) = \dim J(\underline{m})/J(\underline{m}+\underline{e})$  of the homogeneous pieces of the graded ring  $\text{gr}_V R$  can be collected in the Laurent series  $L_V(t_1, \dots, t_r) = \sum_{\underline{m} \in \mathbb{Z}^r} d(\underline{m}) \underline{t}^{\underline{m}}$  (note that the sum extends to  $\mathbb{Z}^r$ ). Following [6] and [11], the Poincaré series of  $V$  is defined as the formal series with integral coefficients

$$P_V(t_1, \dots, t_r) = \frac{L_V(t_1, \dots, t_r) \cdot \prod_{i=1}^r (t_i - 1)}{t_1 t_2 \cdots t_r - 1}.$$

As an application of the results and techniques developed in the previous sections, Section 4 is devoted to the computation of the Poincaré series  $P_V$ . So, in Theorem 6 we state the relation between the Poincaré series of  $V$  and the Poincaré polynomial  $P_C$  of any general curve  $C$  of  $V$ . This polynomial coincides with the Alexander polynomial of the link of the singularity [6]. In the complex case, the above expression leads to an explicit formula for  $P_V$  in terms of the topology of the exceptional divisor, very similar to the formula of A'Campo (see [1]) for the zeta function (extended by Eisenbud and Neumann in [13] for the Alexander polynomial):

$$P_V(t_1, \dots, t_r) = \prod_{E_\alpha \subset E} (1 - \underline{t}^{\underline{\nu}^\alpha})^{-\chi(\dot{E}_\alpha)}$$

where  $\chi(\dot{E}_\alpha)$  is the Euler characteristic of the smooth part  $\dot{E}_\alpha$  of  $E_\alpha \subset E$ . This formula was conjectured by the authors some time ago, but the first complete proof has been given in [11] by using a very different approach: the integration on infinite dimensional spaces with respect to the Euler characteristic.

To prove our results, we develop two different kind of techniques which in our opinion have interest by themselves. The main steps of the first one are included in Section 1. There, we consider pairs of elements in  $R$  with the same value by a valuation  $\nu_\alpha$  associated to a component  $E_\alpha$  of the exceptional divisor  $E$  of a modification and we find the relation between the initial forms of these elements with respect to  $\nu_\alpha$  as well as their values for the valuations corresponding to other components of  $E$ . Such study is given in terms of the topology of the exceptional divisor. In the proofs we systematically use the geometry of pencils of plane curves. In particular, the fact that the divisor  $E_\alpha$  be dicritical for the pencil  $\{\lambda f + \mu g\}$  if and only if the initial forms of the functions  $f$  and  $g$  with respect to  $\nu_\alpha$  are linearly independent, explains the deep relationship between both concepts.

The second technique, specially used in Section 4, is the cited approximation of curves by divisorial valuations. We can see it as a way to go from results related to a curve  $C$  to similar results for the corresponding sets  $V^{(k)}$  of divisorial valuations, and viceversa. Corollary 2, which presents the Poincaré polynomial of a general curve  $C$  of  $V$  as the limit of the Poincaré series of sets of divisorial valuations  $V^{(k)}$ , gives a good example of this philosophy.

## 1 Divisorial valuations

Let  $(R, \mathfrak{m})$  be a local, regular, complete and 2-dimensional ring with an algebraically closed coefficient field  $K$ . For us, a curve will be a subscheme of  $\text{Spec } R$ ,  $C_f$ , defined by some element  $f \in \mathfrak{m}$ . A **divisorial valuation**  $\nu$  is a discrete valuation of the fraction field of  $R$ , centered at  $R$  (i.e.,  $R \cap \mathfrak{m}_\nu = \mathfrak{m}$ , where  $(R_\nu, \mathfrak{m}_\nu)$  is the valuation ring of  $\nu$ ), with rank 1 and transcendence degree 1.

Given a **modification**, that is, a finite sequence of point blowing-ups,  $\pi : X \rightarrow \text{Spec } R$ , there is a divisorial valuation,  $\nu_\alpha$ , associated to each irreducible component  $E_\alpha$  of the total exceptional divisor  $E$  of  $\pi$ , namely, for  $f \in R$ ,  $\nu_\alpha(f)$  is the vanishing order of the function  $f \circ \pi : X \rightarrow K$  along the divisor  $E_\alpha$ . We will say that  $\nu_\alpha$  is the  $E_\alpha$ -valuation.

Assume that  $\pi$  is given by the sequence

$$\pi : X = X_{N+1} \xrightarrow{\pi_{N+1}} X_N \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = \text{Spec } R,$$

and denote, for  $0 \leq i \leq N$ , by  $P_i$  the center of  $\pi_{i+1}$  in  $X_i$  ( $P_0 = \mathfrak{m}$ ), by  $(R_i, \mathfrak{m}_i)$  the local ring of  $X_i$  at  $P_i$ , and by  $E_{i+1}$  the exceptional divisor of  $\pi_i$ . Then, for  $1 \leq \alpha \leq N+1$ ,  $\nu_\alpha$  is the  $\mathfrak{m}_{\alpha-1}$ -adic valuation. Given  $\nu = \nu_\alpha$ , we will sometimes denote  $P_\nu$  and  $E_{\alpha(\nu)}$  instead of  $P_\alpha$  and  $E_\alpha$ .

In fact, divisorial valuations correspond 1 – 1 to finite sequences of point blowing-ups, by associating to  $\nu$  its **minimal resolution**, defined as follows:  $\pi_{i+1}$  is the blowing-up of  $X_i$  at  $P_i$ ,  $P_0 = \mathfrak{m}$ , and for  $i \geq 1$   $P_i$  is the unique point in the exceptional divisor of  $\pi_i$ ,  $E_i$ , such that  $R_\nu$  dominates the local ring of  $X_i$  at  $P_i$ . In this way,  $\nu$  is the divisorial valuation associated to  $E_{N+1}$ .

Given the  $E_\nu$  divisorial valuation  $\nu$ , denote by  $\mathcal{C}_\nu$  the set of all irreducible curves in  $\text{Spec } R$  whose strict transform by the minimal resolution  $\pi$  of  $\nu$  is smooth and meets  $E_\nu$  transversely at a nonsingular point of the total exceptional divisor of  $\pi$ . An element  $f \in \mathfrak{m}$  is said to be a **general element**

of  $\nu$  if  $C_f \in \mathcal{C}_\nu$ . In [18] it is proved that for  $f \in R$ ,

$$\begin{aligned} \nu(f) &= \min\{(f, g) \mid g \in \mathcal{C}_\nu\} \\ &= (f, g) \text{ if } \widetilde{C}_f \cap \widetilde{C}_g = \emptyset \text{ and } g \in \mathcal{C}_\nu, \end{aligned} \quad (1)$$

where  $(f, g)$  stands for the intersection multiplicity  $(C_f, C_g)$  between the curves  $C_f$  and  $C_g$  and  $\widetilde{C}_f, \widetilde{C}_g$  for the strict transforms by  $\pi$  of the curves  $C_f$  and  $C_g$ . The minimal resolution  $\pi : X \rightarrow \mathbf{Spec} R$  of the divisorial valuation  $\nu$  is an embedded resolution of  $C_f$  for  $f \in \mathcal{C}_\nu$ , in general not the minimal one.

Conversely, let  $C_f$  be any irreducible curve in  $\mathbf{Spec} R$ , and take the associated (infinity) sequence of blowing-ups with centers at the infinitely near points of  $f$ ,

$$\cdots \longrightarrow X_{i+1} \xrightarrow{\pi_{i+1}} X_i \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = \mathbf{Spec} R. \quad (2)$$

For each  $i \geq 0$  set  $\nu_i$  the  $E_{i+1}$  divisorial valuation. Since the curve  $C_f$  is determined by the sequence (2), we can think of the sequence of valuations  $\{\nu_i\}_{i \geq 0}$  as an approaching of  $C_f$ . Indeed, for any  $g \in R$ , nonzero in the ring  $R/(f)$ ,  $\nu_i(g) = (f, g)$  for  $i \gg 0$ . So, the study of divisorial valuations and of irreducible curves is closely related (see for example [18]).

Let  $\pi : (X, E) \rightarrow (\mathbf{Spec} R, \mathfrak{m})$  be a modification. The **dual graph**  $\mathcal{G}(\pi)$  of  $\pi$  is the dual figure of the exceptional divisor  $E$ ; that is, it is a graph with a vertex  $\alpha$  for each irreducible component  $E_\alpha$  of  $E$  and where two vertices are adjacent if and only if their corresponding exceptional divisors intersect.

The graph  $\mathcal{G}(\pi)$  is a tree. We will denote by  $\mathbf{1}$  the vertex corresponding to the first exceptional divisor and by  $[\beta, \alpha]$  the path joining  $\beta$  and  $\alpha$ . Along this paper, for a vertex  $\alpha$  in  $\mathcal{G}(\pi)$ ,  $Q_\alpha$  will stand for any irreducible element of  $\mathfrak{m}$  such that the strict transform of the curve  $C_{Q_\alpha}$  on  $X$  is smooth and meets  $E_\alpha$  transversely at a nonsingular point.  $C_{Q_\alpha}$  gives in particular a general element of the  $E_\alpha$ -valuation.

A **dead end** (respectively, **star vertex**) of the graph  $\mathcal{G}(\pi)$  is a vertex which is adjacent to a unique (respectively, to at least three) vertices. The set of dead ends will be denoted by  $\mathcal{E}$ . Given a dead end  $\rho \neq \mathbf{1}$ ,  $st_\rho$  will denote the nearest to  $\rho$  star vertex of  $\mathcal{G}(\pi)$ .

In this paper we will make use of the concept of **pencil** of elements in  $R$ . Recall that if we consider the pencil  $L = \{\lambda f + \mu g \mid \lambda, \mu \in K\}$ , relative to two elements  $f, g \in R$ , a component  $E_\alpha$  of the exceptional divisor  $E$  of a modification  $\pi$  is said to be **dicritical** for  $L$  if the  $E_\alpha$ -valuation,  $\nu_\alpha$ , is constant on  $L$ . This condition is equivalent to say that the lifting  $\widetilde{\varphi} = \varphi \circ \pi$  of the rational function  $\varphi = f/g$  to  $X$  restricts to a surjective (that is, non

constant) morphism from  $E_\alpha$  onto  $\mathbb{P}_K^1$ . In the sequel, we will identify  $\mathbb{P}_K^1$  with  $K \cup \{\infty\}$ . The fibers of  $L$  are studied in [12] in the analytic complex case, and we will use those results because they can be easily extended to our context.

In particular from Theorems 1, 2 and 3 in [12] we can deduce the following:

Let  $\pi : (X, E) \rightarrow \mathbf{Spec} R$  be a modification and  $\alpha$  a vertex of  $\mathcal{G}(\pi)$ . For a subset  $A$  of  $\mathcal{G}(\pi)$  denote  $E_A = \bigcup_{\beta \in A} E_\beta$ . Assume that  $\tilde{\varphi}$  is constant in  $E_\alpha$ ,

$\tilde{\varphi}|_{E_\alpha} \equiv c \in \mathbb{P}^1$ . Then the strict transform  $\tilde{C}_{f-cg}$  of  $C_{f-cg}$  intersects  $E_A$ ,  $A$  being the maximal connected subset of  $\mathcal{G}(\pi)$  such that  $\alpha \in A$ , and  $\tilde{\varphi} \equiv c$  along  $E_A$ .

On the other hand, assume that  $E_\alpha$  is dicritical and  $P \in E_\alpha$  is such that  $\tilde{\varphi}(P) = c$ . If  $P$  is a smooth point of  $E$  then  $\tilde{C}_{f-cg}$  intersects  $E_\alpha$  in  $P$ , and if  $P$  is singular and  $\Delta$  is the connected component of  $\mathcal{G}(\pi) \setminus \{\alpha\}$  such that  $E_\Delta \cap E_\alpha = \{P\}$  then  $\tilde{C}_{f-cg}$  intersects  $E_\Delta$ .

Next results are stated for a modification  $\pi : (X, E) \rightarrow \mathbf{Spec} R$  and a vertex  $\alpha \in \mathcal{G}(\pi)$ .

**Lemma 1** *Let  $h \in R$  be such that  $\nu_\alpha(h) = \nu_\alpha(Q_\alpha)$  and assume that  $\tilde{C}_h \cap \tilde{C}_{Q_\alpha} = \emptyset$ . Then  $E_\alpha$  is the unique dicritical divisor of the pencil  $L = \{\lambda Q_\alpha + \mu h \mid \lambda, \mu \in K\}$ , and the lifting  $\tilde{\varphi}$  of the rational function  $\varphi = Q_\alpha/h$  to  $X$  restricts to an isomorphism in  $E_\alpha$ .*

**Proof.** Since  $\nu_\alpha(Q_\alpha) = \nu_\alpha(h)$ ,  $\tilde{\varphi}$  is defined in every point of  $E_\alpha$ , and  $\tilde{\varphi}_\alpha := \tilde{\varphi}|_{E_\alpha} \not\equiv 0, \infty$ . Since moreover  $\tilde{C}_h \cap \tilde{C}_{Q_\alpha} = \emptyset$ ,  $\tilde{\varphi}_\alpha(\tilde{C}_{Q_\alpha} \cap E_\alpha) = 0$ , so  $E_\alpha$  is a dicritical component for  $L$ . In fact,  $E_\alpha$  is the unique dicritical component for  $L$ , because the existence of another one would contradict the irreducibility of the fiber  $Q_\alpha$ .

Let  $P \in E_\alpha$  be such that  $\tilde{\varphi}_\alpha(P) = 0$ . If there were a connected component  $\Delta$  of  $\mathcal{G}(\pi) \setminus \{\alpha\}$  such that  $P \in E_\Delta$ , then  $\tilde{C}_{Q_\alpha}$  would intersect  $E_\Delta$ , which is impossible by the election of  $Q_\alpha$ . Hence,  $P$  is a smooth point of  $E$ ,  $P = \tilde{C}_{Q_\alpha} \cap E_\alpha$ . Moreover, from Theorem 3 of [12],  $P$  is not a critical point of  $\tilde{\varphi}_\alpha$ , so  $\tilde{\varphi}_\alpha$  has degree 1, i.e., it is an isomorphism.  $\square$

The next result is a generalization of Lemma 4 in [3].

**Proposition 1** *Let  $h \in R$  be such that  $\nu_\alpha(h) = \nu_\alpha(Q_\alpha)$  and such that the strict transform  $\tilde{C}_h$  of  $C_h$  on  $X$  does not intersect  $E_\alpha$ . Then there exists a unique connected component  $\Delta$  of  $\mathcal{G}(\pi) \setminus \{\alpha\}$  such that  $\tilde{C}_h \cap E_\Delta \neq \emptyset$ . Moreover,  $\nu_\gamma(h) = \nu_\gamma(Q_\alpha)$  if  $\gamma \in \mathcal{G}(\pi) \setminus \Delta$ , and  $\nu_\gamma(h) > \nu_\gamma(Q_\alpha)$  otherwise.*



**Proof.** Keep the notations of Lemma 1. Let  $\Delta$  be a connected component of  $\mathcal{G}(\pi) \setminus \{\alpha\}$  such that  $\tilde{C}_h \cap E_\Delta \neq \emptyset$ . By Lemma 1, there are not dicritical divisors of  $L$  in  $\Delta$ , and since  $\tilde{\varphi}(P) = \infty$  for any  $P \in \tilde{C}_h \cap E_\Delta$ , then  $\tilde{\varphi}|_{E_\Delta} \equiv \infty$ , which implies  $\nu_\gamma(h) > \nu_\gamma(Q_\alpha)$  for  $\gamma \in \Delta$ . As  $\tilde{\varphi}_\alpha$  is an isomorphism,  $E_\alpha \cap E_\Delta$  is the unique point  $Q \in E_\alpha$  such that  $\tilde{\varphi}(Q) = \infty$ , hence  $\Delta$  is the unique connected component of  $\mathcal{G}(\pi) \setminus \{\alpha\}$  such that  $\tilde{C}_h \cap E_\Delta \neq \emptyset$  and moreover we deduce that  $\nu_\gamma(Q_\alpha) = \nu_\gamma(h)$  if  $\gamma \notin \Delta$ .  $\square$

A close result holds when we change  $Q_\alpha$  by whatever element  $f \in R$ :

**Proposition 2** *Let  $h$  and  $f$  be elements in  $R$  such that  $\nu_\alpha(h) = \nu_\alpha(f)$  and assume that there exists a connected component  $\Delta$  of  $\mathcal{G}(\pi) \setminus \{\alpha\}$  such that  $E_\Delta$  contains  $\tilde{C}_h \cap E$  and  $\tilde{C}_f \cap E$ . Then  $\nu_\gamma(f) = \nu_\gamma(h)$  for each  $\gamma \notin \Delta$  and there exists  $c \in K$ ,  $c \neq 0$ , such that  $\nu_\alpha(f - ch) > \nu_\alpha(f)$ .*

**Proof.** We can assume that  $\pi$  is an embedded resolution of the curve  $C_{fh}$ , since the additional blowing-ups we need for it do not modify the connected subset  $T = \mathcal{G}(\pi) \setminus \Delta$ .

If  $\nu_\gamma(f) > \nu_\gamma(h)$  for some  $\gamma \in T$ , we deduce that  $\tilde{C}_f$  intersects  $E_A$ , where  $A$  is the maximal connected subset of  $\mathcal{G}(\pi) \setminus \{\alpha\}$  such that  $\gamma \in A$  and  $\nu_\beta(f) > \nu_\beta(h)$  for every  $\beta \in A$ . In particular, as  $A \subset T$ ,  $\tilde{C}_f$  intersects  $E_T$ , which contradicts the hypothesis. Thus,  $\nu_\gamma(f) = \nu_\gamma(h)$  for every  $\gamma \in T$  and the lifting  $\tilde{\varphi}$  of the rational function  $\varphi = f/h$  to the space  $X$  is defined at every point of  $E_T$ . Moreover,  $\tilde{\varphi}|_{E_\beta} : E_\beta \rightarrow \mathbb{P}^1$  cannot be surjective for  $\beta \in T$ , since  $\tilde{C}_h \cap E_T = \emptyset$  and  $\tilde{C}_f \cap E_T = \emptyset$ . Therefore, there exists  $c \in \mathbb{P}_K^1$ ,  $c \neq 0, \infty$ , such that  $\tilde{\varphi}|_{E_T} \equiv c$ . Then the lifting of  $(f - ch)/h$  vanishes on  $E_T$  and in particular  $\nu_\alpha(f - ch) > \nu_\alpha(h)$  (in fact  $\nu_\gamma(f - ch) > \nu_\gamma(h)$  for every  $\gamma \notin \Delta$ ).  $\square$

**Remark.** Let  $f, h \in R$  such that  $\nu_\alpha(f) = \nu_\alpha(h)$  and assume that there exists  $c \in \mathbb{P}^1$ ,  $c \neq 0, \infty$ , such that  $\nu_\alpha(f - ch) > \nu_\alpha(f)$ . Then, the lifting of the rational function  $\varphi = f/h$  is constant and equal to  $c$  along  $E_\alpha$ . As a consequence, the strict transforms of  $C_f$  and  $C_h$  intersect the same points of  $E_\alpha$  and the same connected components of  $\mathcal{G}(\pi) \setminus \{\alpha\}$  (otherwise the corresponding point of intersection in  $E_\alpha$  must be a zero or a pole of  $\varphi$ ).

On the other hand, let  $f \in R$  be such that  $\tilde{C}_f \cap E = P \in E_\alpha$  is a smooth point of  $E$ , set  $r = (E_\alpha, \tilde{C}_f)$  and pick  $Q_\alpha$  by  $P \in E_\alpha$ . Then  $\nu_\alpha(Q_\alpha^r) = \nu_\alpha(f)$  and after some additional blowing-ups we could apply the above proposition, proving the existence of  $c \neq 0, \infty$  such that  $\nu_\alpha(f - cQ_\alpha^r) > \nu_\alpha(f)$ .

Now we recall some known facts about curve singularities and divisorial valuations.



Let  $\nu$  be a divisorial valuation,  $f \in R$  a general element of  $\nu$  defining a curve  $C_f$  and  $v$  the discrete valuation of the fraction field of  $R/(f)$  given by its integral closure. Let  $h$  be an element of  $R$  such that the strict transform  $\tilde{C}_h$  of  $C_h$  by the minimal resolution of  $\nu$  does not intersect the strict transform  $\tilde{C}_f$  of  $C_f$ . Then Equality (1) implies that  $\nu(h) = v(h) = (f, h)$ . If  $\tilde{C}_f \cap \tilde{C}_h \neq \emptyset$  one can use a generic element  $f'$  for which  $\tilde{C}_{f'} \cap \tilde{C}_h = \emptyset$  and  $\nu(h) = v'(h)$ , where  $v'$  is the valuation corresponding to  $f'$ .

The dual graph of the minimal resolution of a valuation  $\nu$  looks like that of Figure 1, where  $\alpha(\nu)$  is the vertex corresponding to the divisor  $E_{N+1} = E_{\alpha(\nu)}$  defining the valuation  $\nu$ ,  $st_i$  stands for the star vertex of the dead end  $\rho_i$  and  $\Gamma_i$  denotes the path from  $st_{i-1}$  to  $\rho_i$ .

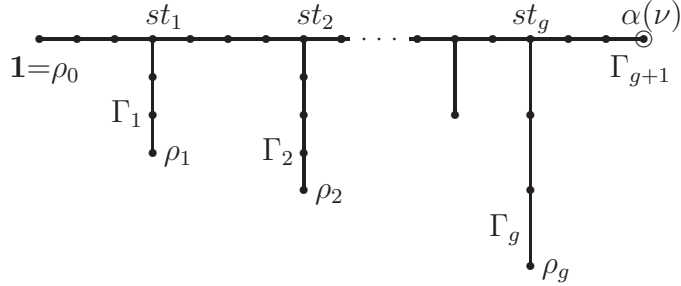


Figure 1: The dual graph of a divisorial valuation.

If  $C_f$  is general for  $\nu$  (i.e.,  $f$  is a general element of  $\nu$ ), then the dead ends of  $\mathcal{G}(\pi)$ ,  $\rho_0, \dots, \rho_g$ , are also dead ends for the dual graph of  $C_f$ , which is the dual graph of the minimal embedded resolution of  $C_f$  together with an arrow attached to the vertex,  $\alpha(f)$ , corresponding to the component intersected by  $\tilde{C}_f$ . We will denote  $Q_i := Q_{\rho_i}$  and we set  $\bar{\beta}_i = v(Q_i)$  ( $0 \leq i \leq g$ ), values which are usually called **maximal contact values** of the curve singularity  $C_f$ . It is known that the set  $\{\bar{\beta}_0, \dots, \bar{\beta}_g\}$  and the Puiseux pairs of  $C_f$ , and hence the equisingularity type of  $C_f$ , are equivalent data (e.g.  $\bar{\beta}_0$  is the multiplicity  $m(f)$  of  $C_f$  at the origin). Moreover,  $\{\bar{\beta}_0, \dots, \bar{\beta}_g\}$  is a minimal set of generators of the semigroup of values  $S_{C_f} := \{v(h) \mid h \in R/(f)^*\}$  of  $C_f$ ,  $R/(f)^*$  denoting the nonzero elements of the ring  $R/(f)$ .

For the divisorial valuation  $\nu$  we have  $\nu(Q_i) = \bar{\beta}_i = v(Q_i)$  and so for the semigroup of values of  $\nu$ ,  $S_\nu := \{\nu(h) \mid h \in R \setminus \{0\}\}$ , one has  $S_\nu = \langle \bar{\beta}_0, \dots, \bar{\beta}_g \rangle = S_{C_f}$ . Thus, arithmetical properties of  $v$  are also true for the valuation  $\nu$  (in [20], the reader can see proofs for the main properties which we will use later in this context). For the sake of completeness we will denote  $\bar{\beta}_{g+1} = \nu(Q_{\alpha(\nu)})$ . It holds that  $\bar{\beta}_{g+1} = e_{g-1}\bar{\beta}_g + c$ , where  $e_{g-1}$  is the smallest positive integer such that  $e_{g-1}\bar{\beta}_g \in \langle \bar{\beta}_0, \dots, \bar{\beta}_{g-1} \rangle$  and  $c \geq 0$  is the number of blowing-ups needed to create  $E_{\alpha(\nu)}$  after the divisor corresponding to  $st_g$  was obtained. Thus,  $\bar{\beta}_{g+1}$  gives an additional datum to the semigroup of

values  $S_\nu$  which permits to recover the dual graph of the divisorial valuation  $\nu$  (see [18]). The element  $\bar{\beta}_{g+1}$  has an expression  $\bar{\beta}_{g+1} = \sum_{j=0}^g \lambda_j \bar{\beta}_j$  with  $\lambda_j \geq 0$  for  $0 \leq j \leq g$ , which is unique if we add some restrictions to the coefficients  $\lambda_j$ . The case  $c = 0$  corresponds to  $\alpha(\nu) = st_g$ , or equivalently, to the case in which  $\mathcal{G}(\pi) \setminus \{\alpha(\nu)\}$  has two connected components, and in this case  $\lambda_g = 0$ , thus,  $\bar{\beta}_{g+1} = e_{g-1} \bar{\beta}_g = \sum_{j=0}^{g-1} \lambda_j \bar{\beta}_j$ .

For simplicity, we will often use the term “monomial” to indicate a monomial in the set  $\{Q_\rho \mid \rho \in \mathcal{E}\}$ , that is, a finite product of the type  $\prod_{\rho \in \mathcal{E}} Q_\rho^{\lambda_\rho}$  with  $\lambda_\rho \in \mathbb{Z}_{\geq 0}$ .

**Proposition 3** *Let  $\pi : X \rightarrow \operatorname{Spec} R$  be a modification. Pick  $\alpha \in \mathcal{G}(\pi)$  and let  $\Delta$  be a connected component of  $\mathcal{G}(\pi) \setminus \{\alpha\}$ . Then, there exists a monomial  $q_\Delta = \prod_{\rho \in \mathcal{E} \cap \Delta} Q_\rho^{\lambda_\rho}$  such that  $\nu_\gamma(q_\Delta) = \nu_\gamma(Q_\alpha)$  if  $\gamma \in \mathcal{G}(\pi) \setminus \Delta$  and  $\nu_\gamma(q_\Delta) > \nu_\gamma(Q_\alpha)$  otherwise.*

**Proof.** We only need to find a monomial  $q_\Delta = \prod_{\rho \in \mathcal{E} \cap \Delta} Q_\rho^{\lambda_\rho}$  such that  $\nu_\alpha(q_\Delta) = \nu_\alpha(Q_\alpha)$ , because it would satisfy  $\tilde{C}_{q_\Delta} \cap E_\Delta \neq \emptyset$ , and then, by Proposition 1, it solves our problem.

Firstly, let us assume that  $\pi : X \rightarrow \operatorname{Spec} R$  is the minimal resolution of  $\nu_\alpha$ . With the above notations,  $\bar{\beta}_{g+1} = \nu_\alpha(Q_\alpha) = \sum_{i=0}^g \lambda_i \bar{\beta}_i$  and we have two possibilities depending whether  $\mathcal{G}(\pi) \setminus \{\alpha\}$  is connected or not. In the first case, the decomposition of  $\bar{\beta}_{g+1}$  provides the monomial  $q_\Delta = \prod_{i=0}^g Q_{\rho_i}^{\lambda_i}$ . Otherwise  $\mathcal{G}(\pi) \setminus \{\alpha\}$  has two connected components; then, if  $[1] \in \Delta$ , we have  $\{\rho_0, \dots, \rho_{g-1}\} = \mathcal{E} \cap \Delta$  and the monomial is  $q_\Delta = \prod_{i=0}^{g-1} Q_{\rho_i}^{\lambda_i}$  (recall that in this case  $\lambda_g = 0$ ), and if  $[1] \notin \Delta$  we have  $\{\rho_g\} = \mathcal{E} \cap \Delta$  and the monomial is  $q_\Delta = Q_{\rho_g}^{\lambda_g}$ .

In general, let us denote by  $\pi' : (Y, F) \rightarrow \operatorname{Spec} R$  the minimal resolution of  $\nu_\alpha$  and let  $\sigma : X \rightarrow Y$  be the composition of the sequence of point blowing-ups which produces  $X$  starting from  $Y$ . We claim that if  $\Omega$  is any connected component of  $\mathcal{G}(\pi) \setminus \mathcal{G}(\pi')$  such that  $\sigma(E_\Omega) = P \in E_\beta$  is a smooth point of  $F$ , then there exists a dead end  $\rho \in \mathcal{E} \cap \Omega$  such that  $\nu_\gamma(Q_\rho) = \nu_\gamma(Q_\beta)$  for any  $\gamma \notin \Omega$ . Indeed, it suffices to choose  $\rho$  as an element of  $\mathcal{E} \cap \Omega$  making minimal the number of blowing-ups needed to obtain it, since for this  $\rho$ , the strict transform of  $Q_\rho$  by  $\pi'$  is smooth and transversal to  $F$  at  $P$ .

Now, if  $\sigma(E_\Delta)$  is a smooth point  $P \in E_\alpha$  of  $F$ , the above construction applied to  $\Delta$  gives  $\rho \in \mathcal{E} \cap \Delta$  such that  $\nu_\alpha(Q_\rho) = \nu_\alpha(Q_\alpha)$ , so we can choose  $q_\Delta = Q_\rho$ .

Otherwise,  $\sigma(E_\Delta) \subset \overline{F \setminus E_\alpha}$ . In this case, if some dead end  $\rho'$  of  $\mathcal{G}(\pi')$  is not a dead end of  $\mathcal{G}(\pi)$ , then there exists a connected component  $\Omega$  of  $\mathcal{G}(\pi) \setminus \mathcal{G}(\pi')$  such that  $\sigma(E_\Omega) = P \in E_{\rho'}$ ,  $P$  a smooth point of  $F$ , and our claim gives a dead end  $\rho$  of  $\mathcal{G}(\pi)$  such that  $\nu_\gamma(Q_\rho) = \nu_\gamma(Q_{\rho'})$  (and  $\rho \in \Delta$  if  $\rho' \in \Delta$ ). Hence, if  $\{\rho'_0, \dots, \rho'_g\}$  are the dead ends of  $\mathcal{G}(\pi')$ , we can find  $\{\rho_0, \dots, \rho_g\}$  in  $\mathcal{E} \cap \Delta$  such that  $\nu_\gamma(Q_{\rho_i}) = \nu_\gamma(Q_{\rho'_i}) = \bar{\beta}_i$  for  $0 \leq i \leq g$ , and the monomial is given as in the case in which  $\pi$  is the minimal resolution.  $\square$

To end this section, assume that  $h \in R$  is irreducible and  $\pi : X \rightarrow \operatorname{Spec} R$  a modification such that the strict transform of the curve  $C_h$  by  $\pi$  only meets one irreducible component, that we will denote  $E_{\alpha(h)}$ , of the exceptional divisor of  $\pi$ .

**Proposition 4** *For any vertex  $\beta \in \mathcal{G}(\pi)$ , there exists a monomial  $q = \prod_{\rho \in \mathcal{E}} Q_\rho^{\lambda_\rho}$  such that  $\nu_\beta(q) = \nu_\beta(h)$  and  $\nu_\gamma(q) \geq \nu_\gamma(h)$  for every  $\gamma \neq \beta$ . Moreover, if  $\beta \neq \alpha(h)$ , then the vertices  $\rho$  such that  $\lambda_\rho \neq 0$  belong to the connected component of  $\alpha(h)$  in  $\mathcal{G}(\pi) \setminus \{\beta\}$ .*

**Proof.** We can choose  $Q_{\alpha(h)}$  through  $P = E_{\alpha(h)} \cap \tilde{C}_h$ . Setting  $r = (E_{\alpha(h)}, \tilde{C}_h)$  we have  $\nu_\beta(Q_{\alpha(h)}^r) = \nu_\beta(h)$  for any  $\beta \in \mathcal{G}(\pi)$  (see the remark after Proposition 2). So, it suffices to obtain  $q$  for the case  $Q_{\alpha(h)}$ , since then  $q^r$  would solve the problem for  $h$ .

Now, the monomial  $q_\Delta$  given in Proposition 3 for any connected component  $\Delta$  of  $\mathcal{G}(\pi) \setminus \{\alpha(h)\}$  such that  $\beta \notin \Delta$ , if it exists, satisfies the requirements of the proposition. Moreover, if  $\beta \neq \alpha(h)$ , then  $\beta \notin \Delta \cup \{\alpha(h)\}$  and this set is a connected subset of  $\mathcal{G}(\pi) \setminus \{\beta\}$ , thus  $\Delta \cup \{\alpha(h)\}$  is contained in a connected component of  $\mathcal{G}(\pi) \setminus \{\beta\}$ .

Otherwise, that is  $\beta$  belongs to every connected component of  $\mathcal{G}(\pi) \setminus \{\alpha(h)\}$ ,  $\alpha(h)$  must be a dead end and we can take  $q = Q_{\alpha(h)}$ .  $\square$

## 2 Semigroup of values

Let  $V = \{\nu_1, \dots, \nu_r\}$  be a finite set of  $r \geq 1$  divisorial valuations and denote by  $\mathbb{Z}_{\geq 0}$  the set of nonnegative integers. The **semigroup of values** of  $V$  is the additive subsemigroup  $S_V$  of  $\mathbb{Z}_{\geq 0}^r$  given by

$$S_V = \{\underline{\nu}(h) := (\nu_1(h), \dots, \nu_r(h)) \mid h \in R \setminus \{0\}\}.$$

The **minimal resolution** of  $V$  is a modification  $\pi : (X, E) \rightarrow (\operatorname{Spec} R, \mathfrak{m})$  such that, for each  $i \in \{1, \dots, r\}$ ,  $\nu_i$  is the  $E_{\alpha(i)}$ -valuation for an irreducible

component  $E_{\alpha(i)}$  of the exceptional divisor  $E$ , and  $\pi$  is minimal with this property. It is clear that a minimal resolution of  $V$  can be recursively obtained by blowing-up  $\text{Spec } R$  at  $\mathfrak{m}$  and any new obtained space  $X_i$  at the closed centers of the valuations in  $V$ . The dual graph of  $V$  is the dual graph of  $\pi$  with the vertices  $\alpha(i)$  highlighted (for example, using a different draw for the point, see Figure 1).

Let  $C = \bigcup_{i=1}^r C_i$  be a reduced curve, with components  $C_1, \dots, C_r$ , defined by an element  $f \in R$ , and denote by  $R/(f)^*$  the set of nonzero divisors of the ring  $R/(f)$ . The **semigroup of values**  $S_C$  of  $C$  is the additive subsemigroup of  $\mathbb{Z}_{\geq 0}^r$  given by

$$S_C := \{ \underline{v}(g) = (v_1(g), \dots, v_r(g)) \mid g \in R/(f)^* \},$$

where each  $v_i$  is the valuation corresponding to  $C_i$ . Sometimes we will consider “the value”  $\underline{v}(h)$  (not in  $S_C$ ) of zero divisors of  $R/(f)$ , understanding  $\nu_i(h) = \infty$  for  $h$  in the ideal of  $R$  defining  $C_i$ , and  $n < \infty$  for any  $n \in \mathbb{Z}_{\geq 0}$ .

The dual graph of  $C$  is the dual graph of its minimal embedded resolution, attaching an arrow, for each irreducible component  $C_i$  of  $C$ , to the exceptional component which meets the strict transform on  $X$  of  $C_i$ . The equisingularity type of  $C$  (i.e., the set of Puiseux pairs for each branch  $C_i$  of  $C$  together with the intersection multiplicities between pairs of branches) and its dual graph, labelling each vertex  $\alpha$  with the minimal number of blowing-ups needed to create  $E_\alpha$ ,  $w(\alpha)$ , are equivalent data.

Let  $\mathcal{G}$  and  $S_V$  be the dual graph and the semigroup of values of a set  $V = \{\nu_1, \dots, \nu_r\}$  of divisorial valuations,  $r > 1$ . A **general curve** of  $V$  is a reduced plane curve with  $r$  branches defined by  $r$  different equations given by general elements of each valuation  $\nu_i$ . An element  $\underline{m} \in S_V$  is said to be **indecomposable** if we cannot write  $\underline{m} = \underline{n} + \underline{k}$  with  $\underline{n}, \underline{k} \in S_V \setminus \{0\}$ .

For  $1 \leq i \leq r$  set  $\alpha(i) = \alpha(\nu_i)$ , and for each vertex  $\rho \in \mathcal{E}$  denote by  $\beta_\rho$  the nearest vertex to  $\rho$  in  $\Omega = \bigcup_{i=1}^r [\mathbf{1}, \alpha(i)]$  (i.e.  $\beta_\rho = \max(\Omega \cap [\mathbf{1}, \rho])$ ). Consider the set

$$\mathcal{H} = \{\mathbf{1}\} \cup \mathcal{E} \cup (\Omega \setminus \{\Gamma \cup \{\beta_\rho \mid \rho \in \mathcal{E}\}\}) ,$$

where  $\Gamma = \bigcap_{i=1}^r [\mathbf{1}, \alpha(i)]$ . Then we can state the following

**Theorem 1** *The set of indecomposable elements of the semigroup of values  $S_V$  is the set  $\{\underline{\nu}(Q_\alpha) \mid \alpha \in \mathcal{H}\}$ . In particular,  $S_V$  is finitely generated.*

This theorem is the divisorial version of the next one which holds for a reduced plane curve  $C$  with  $r$  branches ([4]). In it, we consider the dual graph of  $C = \bigcup_{i=1}^r C_i$  and define  $\mathcal{H}$  as above, and  $\alpha(1), \dots, \alpha(r)$  are the vertices with arrows, corresponding to the branches  $C_1, \dots, C_r$  of  $C$ .

**Theorem 2** *The set of indecomposable elements of the semigroup  $S_C$  is*

$$\{\underline{v}(Q_\alpha) \mid \alpha \in \mathcal{H}\} \cup \{\underline{v}(Q_{\alpha(i)}) + (0, \dots, 0, k, 0, \dots, 0) \mid i = 1, \dots, r \quad k \geq 1\},$$

where  $k$  is in the  $i$ th component.  $\square$

**Proof.** Let us prove Theorem 1. If  $C = \bigcup_{i=1}^r C_i$  is any general curve of  $V$ , that is,  $C_i$  is general for  $\nu_i$ , then  $S_V \subseteq S_C$ , therefore, by Theorem 2, elements in the set  $\{\underline{v}(Q_\alpha) \mid \alpha \in \mathcal{H}\}$  are indecomposable. Conversely, given  $h \in R$  such that  $\underline{v}(h)$  is indecomposable in  $S_V$ , choose a general curve  $C$  of  $V$  such that the strict transforms of  $C$  and  $C_h$  by the minimal resolution of  $V$  do not intersect. So, from equality (1),  $\underline{v}(h) = \underline{v}(h)$  and  $\underline{v}(Q_\alpha) = \underline{v}(Q_\alpha)$  for any vertex  $\alpha$ ,  $\underline{v}$  given by the valuations associated to  $C$ . Moreover,  $h$  must be irreducible and by the proof of Theorem 2, [4],  $\underline{v}(h)$  decomposes in  $S_C$  as a sum of elements  $\underline{v}(Q_\gamma)$  with  $\gamma \in \mathcal{H}$ , which proves that  $\underline{v}(h) = \underline{v}(Q_\alpha)$  for some  $\alpha \in \mathcal{H}$ .  $\square$

**Remark.** A consequence of Theorem 1 is that the semigroup  $S_V$  does not have conductor whenever  $r > 1$ , that is, there is no element  $\delta \in S_V$  such that  $\delta + \mathbb{Z}_{\geq 0}^r \subseteq S_V$ . However, the semigroup of values of a curve with  $r$  branches does have a conductor  $\delta$  [8, Th. 2.7], and thus, it cannot be finitely generated if  $r > 1$ . In particular, if  $C$  is any general curve of  $V$ ,  $S_V \neq S_C$  when  $r > 1$  (recall that  $S_V = S_C$  if  $r = 1$ ).

Considering the ordering over  $\mathbb{Z}^r$  given by  $\underline{n} \leq \underline{m} \Leftrightarrow \underline{m} - \underline{n} \in \mathbb{Z}_{\geq 0}^r$ , a finite set of divisorial valuations  $V = \{\nu_1, \dots, \nu_r\}$  induces a multi-index filtration of the ring  $R$  by means of the valuation ideals  $J(\underline{m})$ ,  $\underline{m} = (m_1, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r$ :

$$J(\underline{m}) := \{g \in R \mid \underline{v}(g) \geq \underline{m}\}.$$

For  $J \subset \{1, \dots, r\}$  denote by  $\underline{e}_J$  the element of  $\mathbb{Z}_{\geq 0}^r$  whose  $i$ th component is equal to 1 (respectively, to 0) if  $i \in J$  (respectively,  $i \notin J$ ); denote  $\underline{e} = \underline{e}_{\{1, \dots, r\}}$ . We will use  $\underline{e}_i$  instead of  $\underline{e}_{\{i\}}$ .

We will denote  $D(\underline{m}) = J(\underline{m})/J(\underline{m} + \underline{e})$  and  $D_i(\underline{m}) = J(\underline{m})/J(\underline{m} + \underline{e}_i)$  for  $1 \leq i \leq r$ . It is clear that the natural homomorphism  $D(\underline{m}) \rightarrow D_1(\underline{m}) \times \dots \times D_r(\underline{m})$  is injective. For  $h \in J(\underline{m}) \setminus J(\underline{m} + \underline{e}_i)$  we will denote  $\text{in}_{\nu_i}(h) =$

$h + J(\underline{m} + \underline{e}_i) \in D_i(\underline{m})$ , and call it **the initial form of  $h$**  with respect to  $\nu_i$ .

When  $r = 1$ , Nakayama's Lemma proves that for any  $m \in \mathbb{Z}$ ,  $D(m)$  is a finite dimensional  $K$ -vector space and, therefore, so are  $D(\underline{m})$  and  $D_i(\underline{m})$  for  $\underline{m} \in \mathbb{Z}_{\geq 0}^r$ . Set  $d(\underline{m}) = \dim D(\underline{m})$  and  $d_i(\underline{m}) := \dim D_i(\underline{m})$ .

In the sequel, we will set  $B^i = \underline{\nu}(Q_{\alpha(i)})$ ,  $i = 1, \dots, r$ . Let  $f \in R$  be such that  $\nu_i(f) = \nu_i(Q_{\alpha(i)})$  (remember that  $\alpha(i)$  denotes the vertex  $\alpha(\nu_i)$  corresponding to the divisor that defines  $\nu_i$ ). Then by Proposition 1,  $\nu_j(f) \geq \nu_j(Q_{\alpha(i)})$  for  $j = 1, \dots, r$ . Moreover, by Lemma 1, if  $\tilde{C}_f \cap \tilde{C}_{Q_{\alpha(i)}} = \emptyset$ , there exists a unique point  $P(f)$  in  $E_{\alpha(i)}$  mapped to  $\infty$  by the lifting of the rational function  $\varphi = Q_{\alpha(i)}/f$ , namely,  $P(f) = \tilde{C}_f \cap E_{\alpha(i)}$  if  $\tilde{C}_f \cap E_{\alpha(i)} \neq \emptyset$  and  $P(f) = E_{\Delta} \cap E_{\alpha(i)}$  if  $\tilde{C}_f \cap E_{\alpha(i)} = \emptyset$  and  $\Delta$  is the connected component of  $G(\pi) \setminus \{\alpha(i)\}$  such that  $\tilde{C}_f \cap E_{\Delta} \neq \emptyset$ . Furthermore, we denote  $P(f) = \tilde{C}_{Q_{\alpha(i)}} \cap E_{\alpha(i)}$  whenever  $\tilde{C}_f \cap \tilde{C}_{Q_{\alpha(i)}} \neq \emptyset$ .

**Proposition 5** *The map  $\Phi : \mathbb{P}D_i(B^i) \rightarrow E_{\alpha(i)}$  from the projectivization of the vector space  $D_i(B^i)$  to the exceptional component  $E_{\alpha(i)}$ , which sends the class  $\text{in}_{\nu_i}(f)$  to  $P(f)$ , is an isomorphism. In particular,  $d_i(B^i) = 2$  and a basis of  $D_i(B^i)$  is given by the initial forms of two elements  $f$  and  $g$  such that  $P(f) \neq P(g)$  (e.g., two  $Q_{\alpha(i)}$  elements at two different points in  $E_{\alpha(i)}$ ).*

**Proof.** First of all, we assert that  $\Phi$  is well-defined. In fact, given  $f, g \in J(B^i) \setminus J(B^i + \underline{e}_i)$  such that  $\text{in}_{\nu_i}(f) = \lambda \text{in}_{\nu_i}(g)$ , that is,  $\nu_i(f - \lambda g) > \nu_i(f) = \nu_i(g) = \nu_i(Q_{\alpha(i)})$  for some  $\lambda \in K \setminus \{0\}$ , the liftings  $\tilde{\varphi}_1$  and  $\tilde{\varphi}_2$  of the rational functions  $f/Q_{\alpha(i)}$  and  $g/Q_{\alpha(i)}$  are defined in  $E_{\alpha(i)}$ , hence the lifting of  $(f - \lambda g)/Q_{\alpha(i)}$  is also defined and it vanishes in  $E_{\alpha(i)}$ . This means that  $\tilde{\varphi}_1 = \lambda \tilde{\varphi}_2$  in  $E_{\alpha(i)}$  and so  $P(f) = P(g)$ .

It is evident that  $\Phi$  is surjective, let us see that it is injective. Take  $f, g \in J(B^i) \setminus J(B^i + \underline{e}_i)$  such that  $P(f) = P(g)$ . If  $\tilde{C}_f \cap \tilde{C}_{Q_{\alpha(i)}} = \emptyset$ , then  $\tilde{C}_g \cap \tilde{C}_{Q_{\alpha(i)}} = \emptyset$ , and, perhaps with some additional blowing-ups, we are in the situation of Proposition 2, so there exists  $\lambda \in K \setminus \{0\}$  such that  $\nu_i(f - \lambda g) > \nu_i(f)$ , that is,  $\text{in}_{\nu_i}(f) = \lambda \text{in}_{\nu_i}(g)$  as we want. Otherwise,  $\tilde{C}_g \cap \tilde{C}_{Q_{\alpha(i)}} \neq \emptyset$  and since  $\nu_i(f) = \nu_i(g) = \nu_i(Q_{\alpha(i)})$ ,  $f, g$  and  $Q_{\alpha(i)}$  are irreducible, smooth and transversal to  $E_{\alpha(i)}$ . Making an additional blowing-up at the point  $P = \tilde{C}_g \cap E_{\alpha(i)} = \tilde{C}_f \cap E_{\alpha(i)}$ , we can conclude, applying again Proposition 2, that  $\text{in}_{\nu_i}(f) = \lambda \text{in}_{\nu_i}(g)$  for some  $\lambda \in K \setminus \{0\}$ .  $\square$

**Proposition 6** *The map  $\tilde{\Phi} : \mathbb{P}D(B^i) \rightarrow E_{\alpha(i)}$  which sends the class of  $f$  to  $P(f)$ , is an isomorphism. In particular  $d(B^i) = 2$ .*

**Proof.** The result is a consequence of the Proposition 5 and of the next lemma.  $\square$

**Lemma 2** *The natural homomorphism  $D(B^i) \rightarrow D_i(B^i)$  is an isomorphism.*

**Proof.** Let  $f \in R$  be such that  $\nu_j(f) \geq B_j^i = \nu_j(Q_{\alpha(i)})$  for every  $j \in \{1, \dots, r\}$  and  $\nu_i(f) > B_i^i = \nu_i(Q_{\alpha(i)})$ . We need to prove that  $\nu_j(f) > B_j^i$  for any  $j$ .

Denote by  $\Delta$  the maximal connected subset of  $\mathcal{G}(\pi)$  such that  $\alpha(i) \in \Delta$  and  $\nu_\beta(f) > \nu_\beta(Q_{\alpha(i)})$  for every  $\beta \in \Delta$ . Notice that the lifting  $\tilde{\varphi}$  of the function  $\varphi = f/Q_{\alpha(i)}$  is defined and it is identically 0 in  $E_\Delta$ , in particular  $E_\beta$  is not dicritical for the pencil  $L = \{\lambda f + \mu Q_{\alpha(i)} \mid \lambda, \mu \in K\}$  for any  $\beta \in \Delta$ . Let us see that  $\Delta = \mathcal{G}(\pi)$ , which proves the lemma.

Otherwise, we could choose a divisor  $E_\beta$  such that  $E_\beta \cap E_\Delta \neq \emptyset$  and  $\beta \notin \Delta$ , that is,  $\nu_\beta(f) \leq \nu_\beta(Q_{\alpha(i)})$ . By making some additional blowing-ups, we can suppose that in fact  $\nu_\beta(f) = \nu_\beta(Q_{\alpha(i)})$ , then  $\tilde{\varphi}$  is defined and it is not constant in  $E_\beta$ , so it is dicritical for  $L$ . Hence, there exists a point  $P \in E_\beta$ ,  $P \notin E_\beta \cap E_\Delta$ , such that  $\tilde{\varphi}(P) = \infty$ , and this means that  $Q_{\alpha(i)}$  meets either  $E_\beta$  at  $P$  or  $E_{\Delta'}$ ,  $\Delta'$  being the connected component of  $P$  in  $\mathcal{G}(\pi) \setminus \{\beta\}$ . But both things are impossible, as  $Q_{\alpha(i)}$  only meets  $E$  at  $E_{\alpha(i)}$ , and  $\alpha(i) \in \Delta \subset \mathcal{G}(\pi) \setminus \Delta'$ .  $\square$

The following two lemmas are devoted to prove Theorem 3 which gives an explicit description of the semigroup  $S_V$  and clarifies the special role of the elements  $B^1, \dots, B^r$ . Fix  $\underline{m} \in \mathbb{Z}^r$  and  $i$  such that  $1 \leq i \leq r$ .

**Lemma 3**  *$d_i(\underline{m}) \geq 2$  if and only if  $d_i(\underline{m} - B^i) \geq 1$ . Moreover, if  $\underline{m} \in S_V$ , then  $d_i(\underline{m}) \geq 2$  if and only if  $\underline{m} - B^i \in S_V$ .*

**Proof.** If  $d_i(\underline{m} - B^i) \geq 1$ , take  $h \in J(\underline{m} - B^i) \setminus J(\underline{m} - B^i + \underline{e}_i)$  and choose a basis  $\{\text{in}_{\nu_i}(h_1), \text{in}_{\nu_i}(h_2)\}$  of  $D_i(B^i)$ . Then  $\text{in}_{\nu_i}(h h_1), \text{in}_{\nu_i}(h h_2)$  are linearly independent vectors in  $D_i(\underline{m})$ .

Conversely, pick  $h_1, h_2 \in J(\underline{m}) \setminus J(\underline{m} + \underline{e}_i)$  whose classes in  $D_i(\underline{m})$  are linearly independent. Every nonzero function of the pencil  $L$  generated by  $h_1$  and  $h_2$ ,  $L = \{\lambda h_1 + \mu h_2 \mid \lambda, \mu \in K\}$ , satisfies  $\nu_i(\lambda h_1 + \mu h_2) = m_i$ , so  $E_{\alpha(i)}$  is dicritical for  $L$ . Therefore, the restriction to  $E_{\alpha(i)}$  of the lifting to  $X$ ,  $\tilde{\varphi}$ , of the rational function  $\varphi = h_1/h_2$  defines a  $s$  to 1 surjective morphism from  $E_{\alpha(i)}$  onto  $\mathbb{P}_K^1$ . Then, a generic fiber  $h = \lambda h_1 + \mu h_2$  of  $L$  can be factorized in  $R$  as  $h = h' \prod_{l=1}^s g_l$ , where the  $g_l$  are irreducible,  $g_l \neq g_j$  when  $l \neq j$  and the strict transform of each curve  $C_{g_l}$  is smooth and transversal



to  $E_{\alpha(i)}$  in a smooth point. Therefore  $\underline{\nu}(g_l) = B^i$  and  $h/g_i \in J(\underline{m} - B^i)$  but  $h/g_i \notin J(\underline{m} - B^i + \underline{e}_i)$ .

Moreover, if  $\underline{m} \in S_V$  then  $h_2$  can be chosen in such a way that  $\underline{\nu}(h_2) = \underline{m}$  and so for  $\lambda$  and  $\mu$  generic we have  $\underline{\nu}(h) = \underline{m}$  and  $\underline{\nu}(h/g_i) = \underline{m} - B^i \in S_V$ .  $\square$

#### Lemma 4

1. If  $\underline{m} \in S_V$  and  $j \neq i$  then  $d_i(\underline{m} + B^j) = d_i(\underline{m})$ .
2. If  $d_i(\underline{m}) \neq 0$  then  $d_i(\underline{m} + B^i) = 1 + d_i(\underline{m})$ .

**Proof.** First, we will prove that if  $j \neq i$  and  $\underline{m} \in S_V$  then the multiplication by  $Q_{\alpha(j)}$  provides a linear bijective map  $\psi : D_i(\underline{m}) \rightarrow D_i(\underline{m} + B^j)$ . Clearly it is injective, let us see that it is also surjective. Pick an element  $f \in R$  such that  $\underline{\nu}(f) = \underline{m}$  and take  $h \in J(\underline{m} + B^j) \setminus J(\underline{m} + B^j + \underline{e}_i)$ . Notice that  $\nu_j(h) \geq \nu_j(fQ_{\alpha(j)})$  and so  $\nu_j(h - \lambda fQ_{\alpha(j)}) \geq \nu_j(fQ_{\alpha(j)})$  for  $\lambda \in K$ .

If  $\nu_j(h - \lambda fQ_{\alpha(j)}) > \nu_j(fQ_{\alpha(j)})$ , for some  $\lambda \in K$ , then there exists an irreducible component  $g$  of  $h$  such that the strict transforms of  $C_g$  and  $C_{Q_{\alpha(j)}}$  by the minimal resolution of  $V$  intersect  $E_{\alpha(j)}$  at the same point, and then  $\text{in}_{\nu_j}(g) = b \cdot \text{in}_{\nu_j}(Q_{\alpha(j)})^c$  for some  $c \geq 1$  and  $b \in K \setminus \{0\}$  (see the remark after Proposition 2). Thus  $h' = bhQ_{\alpha(j)}^c/g$  and  $h$  have the same value and initial form with respect to  $\nu_k$  for  $1 \leq k \leq r$ . In particular,  $\text{in}_{\nu_i}(h) = \text{in}_{\nu_i}(h') \in \text{Im } \psi$ .

Otherwise,  $\nu_j(h - \lambda fQ_{\alpha(j)}) = \nu_j(fQ_{\alpha(j)})$  for all  $\lambda \in K$  and then  $E_{\alpha(j)}$  is a dicritical divisor of the pencil generated by  $h$  and  $fQ_{\alpha(j)}$ . Thus, for a generic  $\lambda$ ,  $h - \lambda fQ_{\alpha(j)}$  has an irreducible component  $g$  such that  $\tilde{C}_g$  is smooth and transversal to  $E_{\alpha(j)}$  at a smooth point. As  $i \neq j$ , by Proposition 2, there exists  $b \in K \setminus \{0\}$  such that  $\text{in}_{\nu_i}(g) = b \text{in}_{\nu_i}(Q_{\alpha(j)})$ . Then  $h' = (h - \lambda fQ_{\alpha(j)})/g \in J(\underline{m}) \setminus J(\underline{m} + \underline{e}_i)$  and  $\text{in}_{\nu_i}(gh') = \text{in}_{\nu_i}(bQ_{\alpha(j)}h') \in \text{Im } \psi$ . Hence  $\text{in}_{\nu_i}(h) = \lambda \text{in}_{\nu_i}(fQ_{\alpha(j)}) + \text{in}_{\nu_i}(gh') \in \text{Im } \psi$ .

Now, we will prove 2. Assume  $j = i$  and pick elements  $h_1, \dots, h_s \in J(\underline{m}) \setminus J(\underline{m} + \underline{e}_i)$  such that the set  $\{\text{in}_{\nu_i}(h_l) | 1 \leq l \leq s\}$  is a basis of  $D_i(\underline{m})$ . Take an irreducible element  $g \in R$  such that  $\tilde{C}_g$  is smooth and transversal to  $E_{\alpha(i)}$  at a smooth point  $P$ ,  $\tilde{C}_g \cap \tilde{C}_{Q_{\alpha(i)}} = \emptyset$  and  $\tilde{C}_g \cap \tilde{C}_{h_s} = \emptyset$ . Then  $\text{in}_{\nu_i}h_1g, \dots, \text{in}_{\nu_i}h_sg, \text{in}_{\nu_i}h_sQ_{\alpha(i)}$  are linearly independent in the vector space  $D_i(\underline{m} + B^i)$ , because in other case we could find  $h = \sum \lambda_i h_i \in J(\underline{m}) \setminus J(\underline{m} + \underline{e}_i)$  and  $\lambda \neq 0$  with  $\nu_i(hg - \lambda h_s Q_{\alpha(i)}) > \nu_i(hg)$  and then  $\tilde{C}_{h_s Q_{\alpha(i)}}$  must intersect  $E_{\alpha(i)}$  at the point  $P$  (see again the remark after Proposition 2), in contradiction with the election of  $g$ . Hence,  $d_i(\underline{m} + B^i) \geq d_i(\underline{m}) + 1$ .

To finish the proof, it suffices to show that if  $d_i(\underline{m} + B^i) = t \geq 2$  then  $d_i(\underline{m}) \geq t - 1$ . In fact, let  $\{\text{in}_{\nu_i}g_1, \dots, \text{in}_{\nu_i}g_t\}$  be a basis of  $D_i(\underline{m} + B^i)$

and consider the family of pencils  $L_k = \{\lambda g_1 + \mu g_k\}$ ,  $2 \leq k \leq t$ . Fix a smooth point  $P \in E_{\alpha(i)}$  in such a way that  $P$  is non-critical for all the pencils  $L_k$ . For each  $k = 2, \dots, t$ , let  $\lambda g_1 + \mu g_k = \varphi_k g'_k$  be the fiber of  $L_k$  corresponding to  $P$  and  $\varphi_k$  the unique irreducible component of such fiber by  $P$ . In this way, all the initial forms of  $\varphi_k$  are equal (up to product by constants). Set  $\mathcal{B} = \{g_1, \varphi_2 g'_2, \dots, \varphi_t g'_t\}$ . Then, for generic  $P$ ,  $\text{in}_{\nu_i}(\mathcal{B})$  is a basis of  $D_i(\underline{m} + B^i)$ , and  $\text{in}_{\nu_i}(g'_2), \dots, \text{in}_{\nu_i}(g'_t) \in D_i(\underline{m})$  are linearly independent elements. Thus  $d_i(\underline{m}) \geq t - 1$  and the proof is finished.  $\square$

**Theorem 3** *For any  $\underline{m} \in S_V$  there exist unique  $a_1, \dots, a_r \in \mathbb{Z}_{\geq 0}$  and  $\underline{n} \in S_V$  such that*

1.  $\underline{m} = \underline{n} + a_1 B^1 + \dots + a_r B^r$ .
2.  $d_i(\underline{n}) = 1$  for every  $i = 1, \dots, r$ .

*In fact  $a_i = \max\{k \in \mathbb{Z}_{\geq 0} \mid \underline{m} - kB^i \in S_V\} = d_i(\underline{m}) - 1$  for  $i = 1, \dots, r$ .*

**Proof.** Assume the existence of the values  $a_1, \dots, a_r, \underline{n}$ , then, by Lemma 4,  $1 = d_i(\underline{n}) = d_i(\underline{m} - a_i B^i) = d_i(\underline{m}) - a_i$ , and by Lemma 3,  $\underline{n} - B^i \notin S_V$ , so  $a_i = \max\{k \in \mathbb{Z}_{\geq 0} \mid \underline{m} - kB^i \in S_V\}$ , and we have the uniqueness. We also have  $a_i = d_i(\underline{m}) - 1$ .

For the existence, define  $a_i = \max\{k \in \mathbb{Z}_{\geq 0} \mid \underline{m} - kB^i \in S_V\}$  and  $\underline{n} = \underline{m} - \sum_k a_k B^k$ . To prove  $\underline{n} \in S_V$  it suffices to prove that if  $\underline{m} - B^i \in S_V$  and  $\underline{m} - B^j \in S_V$  then  $\underline{m} - B^i - B^j \in S_V$ . The conditions  $\underline{m} - B^i \in S_V$  and  $\underline{m} - B^j \in S_V$  imply, by Lemmas 3 and 4 that  $d_j(\underline{m} - B^i) = d_j(\underline{m}) \geq 2$  and hence that  $\underline{m} - B^i - B^j \in S_V$ .  $\square$

**Corollary 1** *Given a modification  $\pi$  and the family of all the valuations associated to the components  $\{E_1, \dots, E_s\}$  of the exceptional divisor of  $\pi$ ,  $W = \{\nu_1, \dots, \nu_s\}$ , it holds that  $S_W = \langle B^1, \dots, B^s \rangle \cong \mathbb{Z}_{\geq 0}^s$ .  $\square$*

**Remark.** The above corollary, established here as a consequence of Theorem 3, was already known, since the determinant of the intersection matrix of the components  $\{E_1, \dots, E_s\}$  of the exceptional divisor of  $\pi$ ,  $M = (E_i \cdot E_j)$ , is  $-1$ , the  $s$  rows of  $A = -M^{-1}$  are exactly the values  $\{B^1, \dots, B^s\}$  and  $S_W = \{\underline{m} \in \mathbb{Z}_{\geq 0}^s \mid -\underline{m}M \geq 0\}$ . Thus the semigroup is the free semigroup generated by the vectors  $B^1, \dots, B^s$ .

In the general case, valuations in  $V$  are those corresponding to a subset  $L$  of  $\{1, \dots, s\}$ ,  $V = V_L = \{\nu_l \mid l \in L\}$ , and then  $S_{V_L}$  is the projection over  $\mathbb{Z}_{\geq 0}^{|L|}$  (that is, over the coordinates in  $L$ ) of the semigroup  $S_W$ , so it is contained in the convex polyhedral cone in  $\mathbb{R}_{\geq 0}^{|L|}$  generated by the elements  $\{B_l \mid l \in L\}$ .

### 3 Graded algebra and generating sequences

Throughout this section, we will consider a nonempty finite set of divisorial valuations  $V = \{\nu_1, \dots, \nu_r\}$  and we will use the notations of the above sections. The **graded  $K$ -algebra** associated to  $V$  is defined to be

$$\mathrm{gr}_V R := \bigoplus_{\underline{m} \in \mathbb{Z}_{\geq 0}^r} \frac{J(\underline{m})}{J(\underline{m} + \underline{e})}.$$

Set  $\Lambda = \{u_j\}_{j \in J}$  a subset of the maximal ideal  $\mathfrak{m}$  of  $R$ . A monomial in  $\Lambda$  is a product  $\prod_{j \in J} u_j^{\gamma_j}$  with  $\gamma_j \in \mathbb{Z}_{\geq 0}$  and  $\gamma_j = 0$  except for a finite subset of  $J$ . Let  $\mathcal{M}(\Lambda)$  denote the set of monomials in  $\Lambda$ , we will say that  $\Lambda$  is a **generating sequence** of  $V$  if for each  $\underline{m} \in \mathbb{Z}_{\geq 0}^r$  the ideal  $J(\underline{m})$  is generated by  $\mathcal{M}_{\underline{m}}(\Lambda) := \mathcal{M}(\Lambda) \cap J(\underline{m})$ . In particular,  $\Lambda$  is a system of generators of  $\mathfrak{m}$ .

A generating sequence  $\Lambda$  of  $V$  is said to be minimal whenever each proper subset of  $\Lambda$  fails to be a generating sequence. In this case  $V$  is said to be **monomial with respect to  $\Lambda$** . Generating sequences of a family  $V$  and its graded algebra  $\mathrm{gr}_V R$  are closely related, as the following result (proved in [7] in a more general context) shows:

**Theorem 4** *Assume that there exists a finite generating sequence for some valuation of  $V$ . Then, a system of generators  $\Lambda = \{u_j\}_{j \in J}$  of the maximal ideal  $\mathfrak{m}$  is a generating sequence of  $V$  if and only if the  $K$ -algebra  $\mathrm{gr}_V R$  is generated by the set  $\bigcup_{j \in J} [u_j]$ , where  $[u_j]$  denotes the cosets that  $u_j$  defines in  $\mathrm{gr}_V R$ .  $\square$*

It is convenient to clarify the sense of the notation  $[u]$  in the above theorem: if  $u \in \mathfrak{m}$  and  $\underline{m} = \underline{\nu}(u)$ , then  $u \in J(\underline{n})$  for any  $\underline{n} \leq \underline{m}$ . Denote  $[u]_{\underline{n}} := u + J(\underline{n} + \underline{e})$ . So,  $[u]_{\underline{n}} \neq 0$  if, and only if,  $\underline{n} + \underline{e} \not\leq \underline{m}$  (that is,  $n_i = m_i$  for some index  $i \in \{1, \dots, r\}$ ). Then,  $[u]$  in Theorem 4 means  $[u] := \{[u]_{\underline{n}} \mid \underline{n} \leq \underline{m} \text{ and } \underline{n} + \underline{e} \not\leq \underline{m}\}$ .

Denote by  $\mathcal{E}$  the set of dead ends of the dual graph of  $V$  and fix an element  $Q_\rho \in R$  for each  $\rho \in \mathcal{E}$ . Set

$$\Lambda_{\mathcal{E}} = \{Q_\rho \mid \rho \in \mathcal{E}\}.$$

Next result is the analogous of Proposition 4 for initial forms of elements in  $R$ .

**Proposition 7** *Given  $h \in R$  and  $i \in \{1, \dots, r\}$ , there exists a linear combination of monomials  $q = \sum a_\lambda q^\lambda$ ,  $q^\lambda = \prod_{\rho \in \mathcal{E}} Q_\rho^{\lambda_\rho}$ , such that  $\nu_i(q) = \nu_i(h)$ ,  $\nu_i(h - q) > \nu_i(h)$  and  $\nu_j(q) \geq \nu_j(h)$  for every index  $j \neq i$ .*

**Proof.** Note that the condition  $\nu_i(h - q) > \nu_i(h)$  is equivalent to  $\text{in}_{\nu_i}(q) = \text{in}_{\nu_i}(h)$ . Thus, it suffices to prove the result for  $h$  irreducible. Let  $\pi$  be the minimal modification such that  $\pi$  is a resolution of  $V$  and the strict transform  $\tilde{C}_h$  of  $C_h$  by  $\pi$  only meets one irreducible component of the exceptional divisor of  $\pi$ ,  $E_{\alpha(h)}$ .

If  $\alpha(h) \neq \alpha(i)$  then, by Proposition 2, there exists  $\lambda$  such that  $\lambda q$ ,  $q$  being the monomial constructed in Proposition 4, satisfies the result. Assume that  $\alpha(h) = \alpha(i)$  and choose  $Q_{\alpha(h)}$  such that  $\tilde{C}_{Q_{\alpha(h)}}$  goes through the intersection point  $E_\alpha \cap \tilde{C}_h$ . Denoting  $m = (E_{\alpha(h)}, \tilde{C}_h)$ , we have  $\nu_j(h) = \nu_j(Q_\alpha^m)$  for  $1 \leq j \leq r$ , and  $\text{in}_{\nu_i}(h) = \lambda \text{in}_{\nu_i}(Q_\alpha^m)$  for some  $\lambda \in K \setminus \{0\}$  (see the remark after Proposition 2), so we only need to prove the statement for  $h = Q_{\alpha(i)}$ .

Let  $\Delta_0, \dots, \Delta_s$  be the connected components of  $\mathcal{G}(\pi) \setminus \{\alpha(i)\}$  and  $q_{\Delta_i}$ ,  $0 \leq i \leq s$  the monomial constructed in Proposition 3 for  $\Delta_i$ . If  $s \geq 1$ , by Proposition 5, the classes of any pair  $q'$  and  $q''$  of such monomials are a basis of  $D_i(\underline{\nu}(Q_{\alpha(i)}))$ , thus  $\text{in}_{\nu_i}(h) = \lambda \text{in}_{\nu_i}(q') + \mu \text{in}_{\nu_i}(q'')$  for some  $\lambda, \mu \in K$  and the linear combination  $q = \lambda q' + \mu q''$  satisfies the requirements of the statement. Finally, if  $s = 0$ , the vertex  $\alpha(i)$  is an end vertex and we can use  $Q_{\alpha(i)}$  together with  $q_{\Delta_0}$  to have a basis of  $D_i(\underline{\nu}(Q_{\alpha(i)}))$ .  $\square$

Now, let  $C$  be a reduced plane curve with  $r$  branches,  $C_1, \dots, C_r$ , and local ring  $\mathcal{O} = R/(f)$ , and denote  $\underline{v} := (v_1, \dots, v_r)$ , where  $v_i$  is the valuation associated to  $C_i$ . We will say that  $\Lambda \subset \mathfrak{m}$  is a generating sequence of  $C$  if the valuation ideals  $J^C(\underline{m}) = \{g \in \mathcal{O} \mid \underline{v}(g) \geq \underline{m}\}$  are generated by the images in  $\mathcal{O}$  of the monomials in  $\Lambda$ . We will set  $c(\underline{m}) := \dim C(\underline{m})$ , where  $C(\underline{m}) = \frac{J^C(\underline{m})}{J^C(\underline{m} + \underline{e})}$  is the corresponding vector space of initial forms. Finally, we define the graded  $K$ -algebra of  $\mathcal{O}$  as

$$\text{gr}\mathcal{O} := \bigoplus_{\underline{m} \in \mathbb{Z}_{\geq 0}^r} \frac{J^C(\underline{m})}{J^C(\underline{m} + \underline{e})}.$$

Denote by  $\mathcal{E}$  the set of dead ends of the dual graph of  $C$  and let  $f_i$  be an element in  $R$  that gives an equation for  $C_i$ , ( $1 \leq i \leq r$ ). We define

$$\Lambda_{\overline{\mathcal{E}}} = \{Q_\alpha \mid \alpha \in \mathcal{E}\} \cup \{f_1, \dots, f_r\}.$$

where we do not include  $f = f_1$  if  $r = 1$ .

Reduced curves can be approached by finite sets of divisorial valuations. Indeed, denote by  $\pi^{(0)} : X^{(0)} \rightarrow \mathbf{Spec} R$  the minimal embedded resolution of the curve  $C$ , and by  $\pi^{(k)} : (X^{(k)}, E^{(k)}) \rightarrow \mathbf{Spec} R$  the composition of  $\pi^{(k-1)}$  with  $r$  additional blowing-ups, one at each point where the strict transform of  $C$  intersects  $E^{(k-1)}$ . Set  $\nu_i^{(k)}$  the  $E_{\alpha^{(k)}(i)}$ -valuation,  $E_{\alpha^{(k)}(i)}$  being the irreducible component of the exceptional divisor  $E^{(k)}$  intersected by the strict transform of the branch  $C_i$ . Then, the sequence  $V^{(k)} = \{\nu_1^{(k)}, \dots, \nu_r^{(k)}\}$  approaches  $C$ , in the sense that for any element  $h \in R$  which is not divisible by any  $f_i$ ,  $\nu_i^{(k)}(h) = (f_i, h) = v_i(h)$  for  $k \gg 0$  and  $1 \leq i \leq r$ .

For  $1 \leq i \leq r$ , denote by  $\alpha^{(0)}(i)$  the vertex of  $\mathcal{G}(\pi^{(0)})$  such that the strict transform of the branch  $C_i$  meets  $E_{\alpha^{(0)}(i)}$ . Then, for any  $k > 0$ , the graph of  $V^{(k)}$ ,  $\mathcal{G}(\pi^{(k)})$ , is obtained by adding  $r$  vertices  $\alpha^{(k)}(1), \dots, \alpha^{(k)}(r)$  (corresponding to the components  $E_{\alpha^{(k)}(i)}$ ) to  $\mathcal{G}(\pi^{(k-1)})$ , each  $\alpha^{(k)}(i)$  adjacent to  $\alpha^{(k-1)}(i)$ . Denoting by  $\mathcal{E}^{(k)}$  the set of dead ends of  $\mathcal{G}(\pi^{(k)})$ , it is clear that  $\mathcal{E}^{(k)} = (\mathcal{E}^{(k-1)} \setminus \{\alpha^{(k-1)}(1), \dots, \alpha^{(k-1)}(r)\}) \cup \{\alpha^{(k)}(1), \dots, \alpha^{(k)}(r)\}$  for  $k \geq 1$ .

Moreover, for each  $k \geq 0$ , the strict transform by  $\pi^{(k)}$  of the branch  $C_i$  is smooth and meets  $E_{\alpha^{(k)}(i)}$  transversally at a nonsingular point, so we can choose  $Q_{\alpha^{(k)}(i)} = f_i$ . In this way, for every  $k \geq 1$ , when  $r > 1$  we have  $\Lambda_{\mathcal{E}^{(k)}} = \Lambda_{\overline{\mathcal{E}}}$ , and  $\Lambda_{\mathcal{E}^{(k)}} = \Lambda_{\overline{\mathcal{E}}} \cup \{f_1\}$  in the case  $r = 1$ .

Note that  $\mathcal{G}(\pi^{(k)}) \subset \mathcal{G}(\pi^{(k+1)})$ , and the (infinite) graph obtained by blowing-up every infinitely near point of  $C$ , is exactly the union  $\bigcup_{k \geq 0} \mathcal{G}(\pi^{(k)})$ .

Analogously, if  $S_{V^{(k)}}$  denotes the value semigroup of the set  $V^{(k)}$ , one gets the inclusion chain  $S_{V^{(0)}} \subseteq S_{V^{(1)}} \subseteq \dots$  and the equality  $S_C = \bigcup_{k \geq 0} S_{V^{(k)}}$ .

**Theorem 5** *Let  $V$  and  $C$  be as above. Then,  $\Lambda_{\mathcal{E}}$  ( $\Lambda_{\overline{\mathcal{E}}}$ , respectively) is a minimal generating sequence of  $V$  ( $C$ , respectively).*

**Proof.** Let us prove first the result for the curve  $C$ . Consider the sequence  $V^{(k)}$  as explained above, in such a way that  $\Lambda_{\mathcal{E}^{(k)}} = \Lambda_{\overline{\mathcal{E}}}$  for every  $k$  if  $r > 1$  and  $\Lambda_{\overline{\mathcal{E}}} = \Lambda_{\mathcal{E}^{(k)}} \setminus \{f_1\}$  if  $r = 1$ . Let  $h \in R$  be such that  $h \notin (f)$ . We claim that there exists a monomial  $q_1$  in  $\Lambda_{\overline{\mathcal{E}}}$  such that  $v_1(h) = v_1(q_1)$  and  $v_i(h) \leq v_i(q_1)$  for  $i = 1, \dots, r$ . To prove the claim it is enough to find such a monomial for each irreducible component of  $h$ , so assume  $h$  irreducible. Moreover, if  $h = f_i$  for some  $i$ , then we can take  $q_1 = f_i$ . Otherwise, take  $k \gg 0$  such that the strict transform of  $C_h$  by  $\pi^{(k)}$  does not intersect any of the components  $E_{\alpha^{(k)}(i)}$ ,  $1 \leq i \leq r$ . Then,  $\nu_i^{(k)}(h) = v_i(h)$  for  $i = 1, \dots, r$ , and applying Proposition 4 to the set  $V^{(k)}$  we find a monomial  $q_1 = \prod_{\rho \in \mathcal{E}^{(k)}} Q_{\rho}^{\lambda_{\rho}}$

such that  $\nu_1^{(k)}(h) = \nu_1^{(k)}(q_1)$ ,  $\nu_i^{(k)}(h) \leq \nu_i^{(k)}(q_1)$  for  $i = 1, \dots, r$  and  $\lambda_{\alpha^{(k)}(1)} = 0$ . In particular,  $f_1 = Q_{\alpha^{(k)}(1)}$  does not appear in the expression of  $q_1$ , so

$q_1$  is a monomial in  $\Lambda_{\overline{\mathcal{E}}}$  even in case  $r = 1$ . Moreover,  $\nu_1^{(k)}(q_1) = v_1(q_1)$  and  $\nu_i^{(k)}(q_1) \leq v_i(q_1)$  for any  $i$ , therefore  $v_1(h) = v_1(q_1)$  and  $v_i(h) \leq v_i(q_1)$  for  $i = 1, \dots, r$ .

So, for  $h \in R \setminus (f)$  we have the monomial  $q_1$  of the claim, and there exists a nonzero constant  $a_1$  with  $v_1(h - a_1 q_1) > v_1(h)$  and  $v_i(h - a_1 q_1) \geq v_i(h)$  for  $i \geq 2$ . The same claim can be applied to an index (if it exists)  $i \geq 2$  such that  $v_i(h - a_1 q_1) = v_i(h)$  and the element  $h - a_1 q_1$ , and iteratively we find a linear combination of monomials  $p = \sum a_i q_i$  satisfying  $\underline{v}(h - p) \geq \underline{v}(h) + \underline{e}$ . Now, if  $\mathcal{E} \neq \emptyset$  choose any  $\rho \in \mathcal{E}$  and set  $Q = Q_\rho$ , and if  $\mathcal{E} = \emptyset$  (in particular  $r \geq 2$ ) choose generic  $\lambda_1, \dots, \lambda_r$  in  $K^*$  and set  $Q = \lambda_1 f_1 + \dots + \lambda_r f_r$ . Repeating the above procedure the times we need, we can finally obtain a finite linear combination  $q$  of monomials in  $\mathcal{M}(\Lambda_{\overline{\mathcal{E}}})$  such that  $\underline{v}(h - q) \geq \delta + \underline{v}(Q)$  where  $\delta$  is the conductor of the semigroup  $S_C$ . The element  $g = (h - q)/Q$  of the total ring of fractions of  $\mathcal{O}$  has value  $\underline{v}(g) \geq \delta$ , in particular it belongs to the integral closure  $\overline{\mathcal{O}}$  of the ring  $\mathcal{O}$  in its total ring of fractions (since  $\overline{\mathcal{O}}$  is in fact the set of elements  $\phi$  of the total ring of fractions such that  $v_i(\phi) \geq 0$  for all  $i = 1, \dots, r$ ). Moreover, the conductor ideal of  $\overline{\mathcal{O}}$  in  $\mathcal{O}$  coincides with the valuation ideal  $J^C(\delta)$ , so  $g \in \mathcal{O}$  and then  $h = q + gQ$  belongs to the ideal generated by  $\mathcal{M}_{\underline{v}(h)}(\Lambda_{\overline{\mathcal{E}}})$ . Thus, the set  $\Lambda_{\overline{\mathcal{E}}}$  is a finite generating sequence for the plane curve  $C$ .

Now, we prove the theorem for the set  $V = \{\nu_1, \dots, \nu_r\}$ . The case  $r = 1$  is proved in [18], hence, by Theorem 4, it suffices to show that for any  $h \in R$ , one can find a linear combination of monomials  $q$  in  $\Lambda_{\mathcal{E}}$  such that  $\nu_i(h - q) > \nu_i(h)$  for all  $i = 1, \dots, r$ . Proposition 7, applied recursively for  $i = 1, \dots, r$ , gives a finite sequence of polynomials  $q_1, \dots, q_r$  in  $\Lambda_{\mathcal{E}}$  such that  $\nu_j(h - \sum_{k=1}^i q_k) > \nu_j(h)$  for  $j \leq i$  and  $\nu_j(h - \sum_{k=1}^i q_k) \geq \nu_j(h)$  for  $j = 1, 2, \dots, r$ .

Hence,  $q = \sum_{k=1}^r q_k$  satisfies our requirements.

To prove the minimality, it is enough to check that any generating sequence must have an element of type  $Q_\rho$  (that is, irreducible and with strict transform smooth and transversal to  $E_\rho$  at a nonsingular point) for each  $\rho \in \mathcal{E}$ .

Suppose  $r = 1$ , and consider the minimal set of generators of the semigroup  $S_C$  or  $S_V$ ,  $\bar{\beta}_0, \dots, \bar{\beta}_g$  (corresponding to  $\mathcal{E} = \{\rho_0, \dots, \rho_g\}$ ). In order to generate  $J(\bar{\beta}_i)$  ( $0 \leq i \leq g$ ), we need at least an element  $h \in R$  such that  $v_1(h) = \nu_1(h) = \bar{\beta}_i$ . But it is known (see e.g. [8]) that in this case  $h$  must be of type  $Q_{\rho_i}$ . In the divisorial case, if  $\alpha(1)$  is a dead end, moreover we have to consider  $\bar{\beta}_{g+1} = \nu_1(Q_{\alpha(1)})$  and from Proposition 5 we deduce that to generate  $J(\bar{\beta}_{g+1})$  we need some element of type  $Q_{\alpha(1)}$ , since all the elements  $h \in R$  such that  $\nu_1(h) = \bar{\beta}_{g+1}$  and  $\tilde{C}_h \cap E_{\alpha(1)} = \emptyset$  have the same initial form.

Now, assume  $r > 1$ . Notice that for any  $W \subset V$  and  $\underline{m}' \in S_W$ ,  $J(\underline{m}') = J(\min pr_W^{-1}(\underline{m}'))$ , where  $pr_W : S_V \rightarrow S_W$  is the projection map. Thus, any generating sequence for  $V$  is also a generating sequence for  $W$  and in particular for  $\nu_i$ ,  $1 \leq i \leq r$ . On the other hand, if  $\rho \in \mathcal{G}(\pi)$  is a dead end of the minimal resolution  $\pi$  of  $V$ , then there exists  $i \in \{1, \dots, r\}$  such that  $\rho$  is a dead end of the minimal resolution of  $\nu_i$ . Therefore we cannot delete any  $Q_\rho$  in our generating sequence and a similar argument holds for curves. Finally, in this last case, if  $v_j(h) = v_j(f_i)$ ,  $j \neq i$  and  $h \neq f_i$ , then  $v_i(h) < k$  for some positive integer  $k$  and  $\underline{m} = (v_1(f_i), \dots, k, \dots, v_r(f_i))$ , where  $k$  is in the  $i$ th coordinate, belongs to  $S_C$ ; hence no  $f_i$  can be omitted to generate  $J^C(\underline{m})$ .  $\square$

**Remark.** We have also proved that minimal generating sequences for  $V$  and  $C$  must be of the form given in Theorem 5. On the other hand, Theorem 4 is also true for the case of curves, so  $[\Lambda_{\overline{\mathcal{E}}}] = \{[Q_\rho]\} \cup \{[f_1], \dots, [f_r]\}$  is a set of generators of  $\text{gr}\mathcal{O}$ . However, here the set  $[\Lambda_{\overline{\mathcal{E}}}]$  has **infinitely many elements**, because  $[f_i]_{\underline{n}} \neq 0$  for infinitely many elements  $\underline{n} \in S_C$ , since  $v_i(f_i) = \infty$ .

## 4 Poincaré series

Along this section we will suppose  $r > 1$ . Let  $\mathcal{L} := \mathbb{Z}[[t_1, t_1^{-1}, \dots, t_r, t_r^{-1}]]$  be the set of formal Laurent series in  $t_1, \dots, t_r$  and  $\underline{t}^{\underline{m}} := t_1^{m_1} \cdot \dots \cdot t_r^{m_r}$  for  $\underline{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$ .  $\mathcal{L}$  is not a ring, but it is a  $\mathbb{Z}[t_1, \dots, t_r]$ -module and a  $\mathbb{Z}[t_1, t_1^{-1}, \dots, t_r, t_r^{-1}]$ -module. For a reduced plane curve  $C$  with  $r$  branches, the formal Laurent series

$$L_C(t_1, \dots, t_r) = \sum_{\underline{m} \in \mathbb{Z}^r} c(\underline{m}) \cdot \underline{t}^{\underline{m}} \in \mathcal{L}$$

was introduced in [6], where it was shown that

$$P'_C(t_1, \dots, t_r) = L_C(t_1, \dots, t_r) \cdot \prod_{i=1}^r (t_i - 1)$$

is in fact a polynomial that is divisible by  $t_1 \cdots t_r - 1$ . The Poincaré series for the curve  $C$  was defined as the polynomial with integer coefficients

$$P_C(t_1, \dots, t_r) = \frac{P'_C(t_1, \dots, t_r)}{t_1 \cdots t_r - 1}.$$

Analogously, for a set of divisorial valuations  $V = \{\nu_1, \dots, \nu_r\}$ , we define

$$L_V(t_1, \dots, t_r) = \sum_{\underline{m} \in \mathbb{Z}^r} d(\underline{m}) \cdot \underline{t}^{\underline{m}} \in \mathcal{L}.$$



$L_V$  is a Laurent series, but, since  $d(\underline{m})$  can be positive even if  $\underline{m}$  have some negative component  $m_i$ , it is not a power series (in fact, as in the case of a curve, it contains infinitely many terms with negative powers). In Proposition 8, we will show that

$$P'_V(t_1, \dots, t_r) = L_V(t_1, \dots, t_r) \cdot \prod_{i=1}^r (t_i - 1) \in \mathbb{Z}[[t_1, \dots, t_r]] .$$

Thus, we define the **Poincaré series** of  $V$  as the formal power series with integer coefficients

$$P_V(t_1, \dots, t_r) = \frac{P'_V(t_1, \dots, t_r)}{t_1 \cdots t_r - 1} .$$

**Remark.** It is not clear a priori whether the Laurent series  $L_V$  can be computed from the Poincaré series  $P_V$ , since in  $\mathcal{L}$  there are elements which vanish after multiplication by  $\prod_{i=1}^r (t_i - 1)$ . So, it is not obvious how to recover neither the Hilbert function of the graded ring  $\text{gr}_V R$ ,  $d(\underline{m})$ ,  $\underline{m} \in \mathbb{Z}_{\geq 0}^r$ , nor the Hilbert function of the multi-index filtration of the ring  $R$ ,  $h(\underline{m}) := \dim R/J(\underline{m})$ ,  $\underline{m} \in \mathbb{Z}_{\geq 0}^r$ . This can be done, following [11], as follows: denote  $I = \{1, \dots, r\}$ , and define

$$\tilde{L}_V(t_1, \dots, t_r) = \sum_{\underline{m} \in \mathbb{Z}_{\geq 0}^r} d(\underline{m}) \cdot \underline{t}^{\underline{m}} \in \mathbb{Z}[[t_1, \dots, t_r]] ,$$

and  $\tilde{P}'_V(t_1, \dots, t_r) = \tilde{L}_V(t_1, \dots, t_r) \cdot \prod_{i=1}^r (t_i - 1)$ . The formula

$$\tilde{P}'_V(t_1, \dots, t_r) = \sum_{J \subset I} (-1)^{\#J} P'_V(t_1, \dots, t_r)|_{\{t_i=1 \text{ for } i \in J\}} ,$$

( $\#J$  denoting the cardinality of  $J$ ) allows to determine the series  $\tilde{P}'_V$  from the series  $P'_V$ , and as a consequence the power series  $\tilde{L}_V$ . Finally,  $\tilde{L}_V$  determines the Laurent series  $L_V$ , since  $d(\underline{m}) = d(\max(m_1, 0), \dots, \max(m_r, 0))$  for  $\underline{m} \not\leq -\underline{1}$  and  $d(\underline{m}) = 0$  for  $\underline{m} \leq -\underline{1}$ .

To get  $h$ , set  $H(t_1, \dots, t_r) := \sum_{\underline{m} \in \mathbb{Z}^r} h(\underline{m}) \cdot \underline{t}^{\underline{m}} \in \mathcal{L}$ , where  $h(\underline{m}) = \dim R/J(\underline{m})$  for  $\underline{m} \in \mathbb{Z}^r$  (notice that  $h(\underline{m}) = 0$  if  $\underline{m} \leq 0$ ). The equality  $H(t_1, \dots, t_r) = L_V(t_1, \dots, t_r)(1 + \underline{t}^{(1, \dots, 1)} + \underline{t}^{(2, \dots, 2)} + \dots)$  solves our problem. Note that the right hand side of the last equality makes sense since  $d(\underline{m}) = 0$  for  $\underline{m} \leq -\underline{1}$ .

The following results involve dimensionality of the homogeneous components of the graded algebra relative to finite sets  $V = \{\nu_1, \dots, \nu_r\}$  of

divisorial valuations. They will be useful to relate the Poincaré series of  $V$  and the one of any general curve for  $V$ . For  $i \in I = \{1, 2, \dots, r\}$ , define

$$\begin{aligned} p_i(\underline{m}) &:= \sum_{J \subset I \setminus \{i\}} (-1)^{\#J} d_i(\underline{m} + \underline{e}_J) \quad \text{and} \\ P_i(t_1, \dots, t_r) &:= \sum_{\underline{m} \in \mathbb{Z}^r} p_i(\underline{m}) \underline{t}^{\underline{m}} \in \mathcal{L}. \end{aligned} \quad (3)$$

**Proposition 8** *Let  $V$  be a finite set of  $r$  divisorial valuations, then*

$$P'_V(t_1, \dots, t_r) = (t_1 t_2 \cdots t_r - 1) P_i(t_1, \dots, t_r) \in \mathbb{Z}[[t_1, \dots, t_r]].$$

*As a consequence  $P_V(t_1, \dots, t_r) = P_i(t_1, \dots, t_r)$  does not depend on the index  $i$  chosen. Moreover, if we write  $P_V(t_1, \dots, t_r) = \sum_{\underline{m} \in \mathbb{Z}^r} p(\underline{m}) \underline{t}^{\underline{m}}$ , then  $\underline{m} \in S_V$  whenever  $p(\underline{m}) \neq 0$  and so  $P_V(t_1, \dots, t_r) \in \mathbb{Z}[[t_1, \dots, t_r]]$ .*

**Proof.** We shall show the first statement for  $i = 1$  for the sake of simplicity. Write  $P'_V(t_1, \dots, t_r) = \sum_{\underline{m} \in \mathbb{Z}^r} \ell(\underline{m}) \underline{t}^{\underline{m}} \in \mathcal{L}$ . Then

$$\begin{aligned} \ell(\underline{m}) &= \sum_{J \subset I} (-1)^{\#J} d(\underline{m} - \underline{e} + \underline{e}_J) \\ &= \sum_{J \subset I \setminus \{1\}} (-1)^{\#J} (d(\underline{m} - \underline{e} + \underline{e}_J) - d(\underline{m} - \underline{e} + \underline{e}_J + \underline{e}_1)). \end{aligned}$$

On the other hand, if  $\underline{n} \in \mathbb{Z}^r$ , then, for any arrangement  $(i_1, \dots, i_r)$  of the elements in the set  $I$ ,  $D(\underline{n}) \simeq \bigoplus_{j=1}^r D_{i_j}(\underline{n} + \underline{e}_{i_1} + \cdots + \underline{e}_{i_{j-1}})$ . So,  $d(\underline{n}) = \sum_{j=1}^r d_{i_j}(\underline{n} + \underline{e}_{i_1} + \cdots + \underline{e}_{i_{j-1}})$ .

Applying the above decomposition for  $\underline{n} = \underline{m} - \underline{e} + \underline{e}_J$  with the natural arrangement  $(1, 2, \dots, r)$  and for  $\underline{n} = \underline{m} - \underline{e} + \underline{e}_J + \underline{e}_1$  with the arrangement  $(2, 3, \dots, r, 1)$  we get:

$$\begin{aligned} \ell(\underline{m}) &= \sum_{J \subset I \setminus \{1\}} (-1)^{\#J} (d(\underline{m} - \underline{e} + \underline{e}_J) - d(\underline{m} - \underline{e} + \underline{e}_J + \underline{e}_1)) \\ &= \sum_{J \subset I \setminus \{1\}} (-1)^{\#J} (d_1(\underline{m} - \underline{e} + \underline{e}_J) - d_1(\underline{m} + \underline{e}_J)) \\ &= p_1(\underline{m} - \underline{e}) - p_1(\underline{m}), \end{aligned}$$

and thus, we obtain the formula for  $P'_V$  given in the statement.

Now, let  $\underline{m} \in \mathbb{Z}^r$  be such that  $\underline{m} \notin S_V$ . Then there exists an index  $i \in I$  such that  $d_i(\underline{m}) = 0$ , since otherwise

$$\underline{m} = \min\{\nu(g_i) \mid \text{in } \nu_i(g_i) \in D_i(\underline{m}) \setminus \{0\}, 1 \leq i \leq r\}$$

is in  $S_V$ .  $D_i(\underline{m} + \underline{e}_J) \subset D_i(\underline{m})$  for any  $J \subset I$  with  $i \notin J$ , thus  $d_i(\underline{m} + \underline{e}_J) = 0$  and so  $p_i(\underline{m}) = 0$ , which ends the proof.  $\square$

We will say that the divisorial valuation  $\nu_j \in V$  is **extremal** if  $\alpha(j)$  is a dead end of  $\mathcal{G}(\pi)$ ,  $\pi$  being the minimal resolution of  $V$ .

**Lemma 5** Take  $\underline{m} \in S_V$  and let  $\nu_j \in V$  be an extremal valuation. Then, for every  $J \subset I$  with  $j \notin J$  the equality  $d_j(\underline{m} + B^j + \underline{e}_J) = d_j(\underline{m} + \underline{e}_J) + 1$  holds. As a consequence,  $p(\underline{m}) = p(\underline{m} + B^j)$ .

**Proof.** Set  $I' = I \setminus \{j\}$  and  $B^j = (B_1^j, \dots, B_r^j)$ . Since  $\nu_j$  is extremal, by Proposition 3 there exists a monomial  $q$  such that  $\nu_j(q) = B_j^j$  and  $\nu_i(q) > B_i^j$  for  $i \in I'$ . Let  $h \in R$  be such that  $\underline{\nu}(h) = \underline{m}$ ; then  $\underline{\nu}(hq) \geq \underline{m} + \underline{e}_{I'} + B^j$  but  $\underline{\nu}(hq) \not\geq \underline{m} + \underline{e} + B^j$ . In particular, for any  $J \subset I'$ ,  $\underline{\nu}(hq) \geq \underline{m} + \underline{e}_J + B^j$  but  $\underline{\nu}(hq) \not\geq \underline{m} + \underline{e}_J + \underline{e}_j + B^j$  and so,  $d_j(\underline{m} + B^j + \underline{e}_J) \neq 0$ .

Consider again  $J \subset I$  such that  $j \notin J$ . If  $d_j(\underline{m} + \underline{e}_J) \neq 0$  then, by Lemma 4,  $d_j(\underline{m} + B^j + \underline{e}_J) = d_j(\underline{m} + \underline{e}_J) + 1$ . If, otherwise,  $d_j(\underline{m} + \underline{e}_J) = 0$ , then  $d_j(\underline{m} + \underline{e}_J + B^j) = 1$ , since  $d_j(\underline{m} + \underline{e}_J + B^j) \geq 2$  implies  $d_j(\underline{m} + \underline{e}_J) \geq 1$  (Lemma 3).

Finally, the fact that  $p(\underline{n}) = p_j(\underline{n})$  for any  $\underline{n} \in \mathbb{Z}^r$  and the following equalities chain conclude the proof (recall that  $r > 1$ )

$$\begin{aligned} p_j(\underline{m} + B^j) &= \sum_{j \notin J \subset I} (-1)^{\#J} d_j(\underline{m} + B^j + \underline{e}_J) \\ &= \sum_{j \notin J \subset I} (-1)^{\#J} (d_j(\underline{m} + \underline{e}_J) + 1) \\ &= p_j(\underline{m}) + \sum_{j \notin J \subset I} (-1)^{\#J} = p(\underline{m}). \quad \square \end{aligned}$$

Let  $C = C_f$  be a general curve of  $V$  and consider the sequence of families of valuations  $\{V^{(k)}\}$  constructed for  $C$  in Section 3, where  $\pi^{(0)}$  is the minimal resolution of  $V$ . Next proposition allows to write the Poincaré series,  $P_{V^{(k)}}(t_1, \dots, t_r)$ , as a quotient of two series in such a way that the numerator does not depend on  $k$ . Stand  $B_{(k)}^i$  for the value  $B^i$  associated to the family  $V^{(k)}$ .

**Proposition 9** For every  $k \geq 0$ ,

$$P_{V^{(k)}}(t_1, \dots, t_r) \cdot \prod_{i=1}^r (1 - \underline{t}^{B_{(k)}^i}) = P_V(t_1, \dots, t_r) \cdot \prod_{i=1}^r (1 - \underline{t}^{B^i}).$$

**Proof.** It suffices to show the formula for  $k = 1$ . Let  $E_{\tilde{\alpha}(1)}$  be the exceptional divisor created by blowing-up at a smooth point  $P \in E_{\alpha(1)}$ . Set  $\tilde{\nu}_1$  the  $E_{\tilde{\alpha}(1)}$  valuation and consider the set of divisorial valuations  $\tilde{V} = \{\tilde{\nu}_1, \nu_2, \dots, \nu_r\}$ . Stand  $P_{\tilde{V}}(t_1, \dots, t_r)$  for the Poincaré series of  $\tilde{V}$  and set  $\tilde{B}^1 = (\tilde{\nu}_1(Q_{\tilde{\alpha}(1)}), \nu_2(Q_{\tilde{\alpha}(1)}), \dots, \nu_r(Q_{\tilde{\alpha}(1)})) \in S_{\tilde{V}}$ . If we prove that

$$(1 - \underline{t}^{\tilde{B}^1}) P_{\tilde{V}}(t_1, \dots, t_r) = (1 - \underline{t}^{B^1}) P_V(t_1, \dots, t_r),$$

then the result, for  $k = 1$ , follows after iterating the same procedure for the remaining valuations  $\nu_i$ .

Let us write  $P_{\tilde{V}}(t_1, \dots, t_r) = \sum_{\underline{m} \in \mathbb{Z}^r} \tilde{p}(\underline{m}) \underline{t}^{\underline{m}}$ . We only need to prove, for any  $\underline{m} \in \mathbb{Z}^r$ , the following equality:

$$\tilde{p}(\underline{m}) - \tilde{p}(\underline{m} - \tilde{B}^1) = p(\underline{m}) - p(\underline{m} - B^1). \quad (4)$$

Indeed, if  $\underline{m} \notin S_V$  (respectively,  $\underline{m} \notin S_{\tilde{V}}$ ) then the right (respectively, the left) hand side of equality (4) vanishes, since both involved terms are equal to zero. Moreover, if  $\underline{m} \in S_{\tilde{V}} \setminus S_V$  then by Theorem 3,  $\underline{m} = \underline{n} + s\tilde{B}^1$  for some  $\underline{n} \in S_V$  and  $s \geq 1$  (since otherwise  $\underline{m} \in S_V$ ). In particular,  $\underline{m} - \tilde{B}^1 \in S_{\tilde{V}}$  and, since  $\tilde{\nu}_1$  is extremal, Lemma 5 implies  $\tilde{p}(\underline{m}) = \tilde{p}(\underline{m} - \tilde{B}^1)$ . Therefore the left hand side of equality (4) is also equal to zero. So, from now on we assume that  $\underline{m} \in S_V$ .

Denote by  $\tilde{J}(\underline{m})$  the valuation ideal of  $\underline{m}$  for  $\tilde{V}$ . Set  $\tilde{d}_1(\underline{m}) = \dim \tilde{J}(\underline{m}) / \tilde{J}(\underline{m} + \underline{e}_1)$ . Taking into account the formulae in (3), to prove (4) we only need to show the following equality for any  $J \subset I \setminus \{1\}$ :

$$\tilde{d}_1(\underline{m} + \underline{e}_J) - \tilde{d}_1(\underline{m} - \tilde{B}^1 + \underline{e}_J) = d_1(\underline{m} + \underline{e}_J) - d_1(\underline{m} - B^1 + \underline{e}_J). \quad (5)$$

Let us assume that either  $d_1(\underline{m} + \underline{e}_J) \neq 0$  or  $\tilde{d}_1(\underline{m} + \underline{e}_J) = 0$ . Since  $S_V \subset S_{\tilde{V}}$ , we have  $d_1(\underline{n}) = 0$  if  $\tilde{d}_1(\underline{n}) = 0$ , for any  $\underline{n} \in \mathbb{Z}^r$ . Therefore, if  $\tilde{d}_1(\underline{m} - B^1 + \underline{e}_J) = 0$ , then  $d_1(\underline{m} - B^1 + \underline{e}_J) = 0$ , and by Lemma 3,  $\tilde{d}_1(\underline{m} + \underline{e}_J) \in \{0, 1\}$  and  $d_1(\underline{m} + \underline{e}_J) \in \{0, 1\}$ . Since our assumption excludes the case  $\tilde{d}_1(\underline{m} + \underline{e}_J) = 1$ ,  $d_1(\underline{m} + \underline{e}_J) = 0$ , the equality (5) holds. Otherwise,  $\tilde{d}_1(\underline{m} - B^1 + \underline{e}_J) \neq 0$ , by Lemma 4 the left hand side of the equality (5) is equal to 1. Again by our assumption, we cannot have  $d_1(\underline{m} + \underline{e}_J) = 0$  and applying Lemma 3 when  $d_1(\underline{m} - B^1 + \underline{e}_J) = 0$  and Lemma 4 otherwise we prove that the right hand side also equals 1.

To finish the proof, we will prove that there is no  $J \subset I \setminus \{1\}$  such that  $d_1(\underline{m} + \underline{e}_J) = 0$  and  $\tilde{d}_1(\underline{m} + \underline{e}_J) \neq 0$ . If  $d_1(\underline{m} + \underline{e}_J) = 0$  and  $\tilde{d}_1(\underline{m} + \underline{e}_J) \neq 0$ , pick  $h \in R$  such that its image in  $\tilde{D}_1(\underline{m} + \underline{e}_J)$  does not vanish. Let  $\tilde{\pi}$  be the minimal resolution of  $\tilde{V}$ . Clearly,  $\mathcal{G}(\tilde{\pi}) \setminus \{\tilde{\alpha}(1)\}$  is connected. Since  $d_1(\underline{m} + \underline{e}_J) = 0$ , we have  $\tilde{\nu}_1(h) \neq \nu_1(h)$ , and as a consequence the strict transform of  $C_h$  by  $\tilde{\pi}$  intersects  $E_{\tilde{\alpha}(1)}$ . Let  $h = \varphi h'$  be such that the strict transform of  $C_{h'}$  by  $\tilde{\pi}$  does not intersect  $E_{\tilde{\alpha}(1)}$  and the strict transforms by  $\tilde{\pi}$  of all the irreducible components of  $C_\varphi$  intersect  $E_{\tilde{\alpha}(1)}$ .

Applying Proposition 3 to the connected component  $\Delta = \mathcal{G}(\tilde{\pi}) \setminus \{\tilde{\alpha}(1)\}$  one can show that there exists a monomial  $q$  in the elements  $\{Q_\rho \mid \rho \in \mathcal{E} \cap \Delta\}$ , such that  $\tilde{\nu}_1(q) = \tilde{\nu}_1(\varphi)$  and  $\nu_i(q) > \nu_i(\varphi)$  for  $i = 2, \dots, r$ . As

the irreducible components of the strict transforms of  $C_q$  do not meet the divisor  $E_{\tilde{\alpha}(1)}$ , one has  $\nu_1(q) = \tilde{\nu}_1(q)$  and so  $\nu_1(h'q) = \tilde{\nu}_1(h'q) = \tilde{\nu}_1(h) = m_1$  and  $\nu_i(h'q) > \nu_i(h) \geq m_i$ . As a consequence,  $h'q \in D_1(\underline{m} + \underline{e}_{I \setminus \{1\}}) \setminus \{0\}$ , and then  $d_1(\underline{m} + \underline{e}_J) \neq 0$ , which is a contradiction.  $\square$

Now, we state the relationship between the Poincaré series of a finite set of divisorial valuations  $V$  and the Poincaré polynomial of a general curve,  $C$ , of  $V$ .

**Theorem 6** *Let  $V$  and  $C$  be as above. Then,*

$$P_V(t_1, \dots, t_r) = \frac{P_C(t_1, \dots, t_r)}{\prod_{i=1}^r (1 - t^{B^i})}.$$

**Proof.** By Proposition 9, it suffices to prove the result for the set of valuations  $V^{(k)}$ , for some  $k$ . In particular, we can assume that all the divisorial valuations  $\nu_1, \dots, \nu_r$  are extremal. Fix some  $k$  and, for simplicity, write  $\tilde{V} = V^{(k)}$ ,  $P_{\tilde{V}}(t_1, \dots, t_r) = \sum_{\underline{m} \in \mathbb{Z}_{\geq 0}^r} \tilde{p}(\underline{m}) \underline{t}^{\underline{m}}$ , and set  $\tilde{d}_i$  for the corresponding dimensions, and recall that  $\tilde{p}(\underline{m}) = 0$  when  $\underline{m} \notin S_{\tilde{V}}$ .

The coefficient of  $\underline{t}^{\underline{m}}$  in the series  $P_{\tilde{V}}(t_1, \dots, t_r) \cdot \prod_{i=1}^r (1 - \underline{t}^{\widetilde{B^i}})$  is

$$\lambda_{\underline{m}} = \sum_{J \subset I} (-1)^{\#J} \tilde{p}(\underline{m} - \sum_{i \in J} \widetilde{B^i}).$$

Now, if  $J_{\underline{m}} = \{i \in I \mid \underline{m} - \widetilde{B^i} \in S_{\tilde{V}}\}$ , we have  $\underline{m} - \sum_{i \in J} \widetilde{B^i} \in S_{\tilde{V}}$  if and only if  $J \subset J_{\underline{m}}$  (see Theorem 3), and in this case, by Lemma 5,  $\tilde{p}(\underline{m} - \sum_{i \in J} \widetilde{B^i}) = \tilde{p}(\underline{m})$ . Therefore,

$$\lambda_{\underline{m}} = \sum_{J \subset J_{\underline{m}}} (-1)^{\#J} \tilde{p}(\underline{m} - \sum_{i \in J} \widetilde{B^i}) = \sum_{J \subset J_{\underline{m}}} (-1)^{\#J} \tilde{p}(\underline{m}),$$

which is 0 if  $J_{\underline{m}} \neq \emptyset$  and  $\tilde{p}(\underline{m})$  if  $J_{\underline{m}} = \emptyset$ , that is, if  $\tilde{d}_i(\underline{m}) = 1$  for  $1 \leq i \leq r$  (Theorem 3). Hence,

$$P_{\tilde{V}}(t_1, \dots, t_r) \cdot \prod_{i=1}^r (1 - \underline{t}^{\widetilde{B^i}}) = \sum_{\underline{m} \in A} \tilde{p}(\underline{m}) \underline{t}^{\underline{m}},$$

where

$$A := \{\underline{m} \in S_{\tilde{V}} \mid \tilde{d}_i(\underline{m}) = 1 \text{ for } 1 \leq i \leq r\}.$$

For  $J \subset I \setminus \{1\}$  we have  $D_1(\underline{m} + e_J) \subset D_1(\underline{m})$ , hence  $\tilde{d}_1(\underline{m} + e_J) \leq 1$  for any  $\underline{m} \in A$ . Thus, all the summands in the formula  $\tilde{p}(\underline{m}) = \tilde{p}_1(\underline{m}) = \sum_{J \subset I \setminus \{1\}} (-1)^{\#J} \tilde{d}_1(\underline{m} + \underline{e}_J)$  are 1 or 0.

On the other hand, if  $P_C(t_1, \dots, t_r) = \sum_{\underline{m}} \bar{p}(\underline{m}) \underline{t}^{\underline{m}}$  is the Poincaré polynomial of the curve  $C$ , it is straightforward to deduce that the coefficients  $\bar{p}(\underline{m})$  satisfy a formula similar to (3), in particular the following equality holds

$$\bar{p}(\underline{m}) = \bar{p}_1(\underline{m}) = \sum_{J \subset I \setminus \{1\}} (-1)^{\#J} c_1(\underline{m} + \underline{e}_J)$$

where  $c_1(\underline{n}) = \dim J^C(\underline{n})/J^C(\underline{n} + e_1)$  for any  $\underline{n} \in \mathbb{Z}^r$ . And in this case, the dimensions  $c_1(\underline{n})$  only could be 1 or 0, because  $J^C(\underline{n})/J^C(\underline{n} + e_1)$  can be regarded as a vector subspace of  $J^{C_1}(\underline{n})/J^{C_1}(\underline{n} + e_1)$  ( $C_1$  being one of the branches of  $C$ ), whose dimension is 1 or 0.

We claim that there exists some  $k$  such that for any  $\underline{m} \in A$  and  $J \subset I \setminus \{1\}$  it happens that  $\tilde{d}_1(\underline{m} + \underline{e}_J) = 0$  if and only if  $c_1(\underline{m} + \underline{e}_J) = 0$ , and such that  $\bar{p}(\underline{m}) = 0$  for any  $\underline{m} \notin A$ . Then, we deduce  $P_{\tilde{V}}(t_1, \dots, t_r) = \frac{P_C(t_1, \dots, t_r)}{\prod_{i=1}^r (1 - t^{B^i})}$ , as we wanted to prove (recall that  $\tilde{V}$  depends on  $k$ ).

Since  $S_{\tilde{V}} \subset S_C$ , then  $\tilde{d}_1(\underline{n}) \neq 0$  implies  $c_1(\underline{n}) \neq 0$  for any  $\underline{n} \in \mathbb{Z}^r$ . Moreover, for  $\underline{m}$  fixed, there exists  $k_0$  such that, for any  $J \subset I \setminus \{1\}$  and  $k \geq k_0$ , we have  $d_1^{(k)}(\underline{m} + \underline{e}_J) \neq 0$  if  $c_1(\underline{m} + \underline{e}_J) \neq 0$ . But  $P_C$  is a polynomial, so  $B := \{\underline{m} \in S_C \mid \bar{p}(\underline{m}) \neq 0\}$  is a finite set and for  $k$  large enough we find  $\tilde{d}_1(\underline{m} + \underline{e}_J) = 0$  if and only if  $c_1(\underline{m} + \underline{e}_J) = 0$ , for any  $J \subset I \setminus \{1\}$  and  $\underline{m} \in B$ ; if moreover we pick  $k$  such that  $\underline{m} \not\geq B_{(k)}^i$  for every  $\underline{m} \in B$ , we have  $B \subset A$  (see Theorem 3), that is,  $\bar{p}(\underline{m}) = 0$  for any  $\underline{m} \notin A$ , which proves our claim.  $\square$

Next corollary gives a precise meaning to the fact that the valuations defined by a curve singularity can be approached by families of divisorial valuations:

**Corollary 2** *Let  $V$  and  $C$  as above and  $V^{(k)}$  ( $k \geq 0$ ) the finite sets of divisorial valuations defined in Section 3. Then,*

$$\lim_{k \rightarrow \infty} P_{V^{(k)}}(t_1, \dots, t_r) = P_C(t_1, \dots, t_r). \quad \square$$

Finally, assume that  $R = \mathcal{O}_{\mathbb{C}^2, 0}$  is the local ring of germs of holomorphic functions at the origin of the complex plane  $\mathbb{C}^2$ . For a vertex  $\alpha$  of the dual graph  $\mathcal{G}$  of a set of valuations  $V$  as above, denote by  $\dot{E}_\alpha = E_\alpha \setminus (\overline{E - E_\alpha})$

the smooth part of an irreducible component  $E_\alpha$  in the exceptional divisor,  $E$ , of the minimal resolution of  $V$  and by  $\chi(\dot{E}_\alpha)$  its Euler characteristic. In addition, set  $\underline{\nu}^\alpha = \underline{\nu}(Q_\alpha)$ . Then the following formula of A'Campo's type [1], firstly proved in [11], holds.

**Corollary 3**

$$P_V(t_1, \dots, t_r) = \prod_{E_\alpha \subset E} (1 - \underline{t}^{\underline{\nu}^\alpha})^{-\chi(\dot{E}_\alpha)}.$$

**Proof.**  $E_\alpha$  is isomorphic to the complex line  $\mathbb{P}_{\mathbb{C}}^1$ , so  $\chi(\dot{E}_\alpha) = 2 - b(\alpha)$ , where  $b(\alpha)$  denotes the number of singular points of  $E_\alpha$  in  $E$  (i.e., the number of connected components of  $\mathcal{G} \setminus \{\alpha\}$ ).

Since the Poincaré polynomial  $P_C(t_1, \dots, t_r)$  coincides with the Alexander polynomial  $\Delta^C(t_1, \dots, t_r)$  (see [6]) and by using the Eisenbud-Neumann formula for  $\Delta^C(t_1, \dots, t_r)$  [13], we obtain:

$$P_C(t_1, \dots, t_r) = \Delta^C(t_1, \dots, t_r) = \prod_{E_\alpha \subset E} (1 - \underline{t}^{\underline{\nu}^\alpha})^{-\chi(\dot{E}_\alpha)},$$

where  $\underline{\nu}^\alpha = \underline{\nu}(Q_\alpha)$ ,  $\underline{\nu}$  being the above described valuation sequence given by  $C$ , and  $\dot{E}_\alpha$  is the smooth part of  $E_\alpha$  in the total transform of  $C$  by the minimal resolution of  $V$ .  $\chi(\dot{E}_\alpha) = 2 - b(\alpha) = \chi(\dot{E}_\alpha)$  for those  $\alpha \notin \{\alpha(1), \dots, \alpha(r)\}$  and  $\chi(\dot{E}_{\alpha(i)}) = 2 - (b(\alpha(i)) + 1) = \chi(\dot{E}_{\alpha(i)}) - 1$  for  $i = 1, \dots, r$ .

Finally,  $\underline{\nu}^\alpha = \underline{\nu}^\alpha$  for  $\alpha \in \mathcal{G}$  and, so,

$$P_V(t_1, \dots, t_r) = \frac{\prod_{E_\alpha \subset E} (1 - \underline{t}^{\underline{\nu}^\alpha})^{-\chi(\dot{E}_\alpha)}}{\prod_{i=1}^r (1 - \underline{t}^{B^i})} = \prod_{E_\alpha \subset E} (1 - \underline{t}^{\underline{\nu}^\alpha})^{-\chi(\dot{E}_\alpha)}. \quad \square$$

**Remark.** The proof of the equality between the Poincaré and the Alexander polynomials in [6] uses the topology of the complex field. However, the authors have informed us about the existence of a non-published alternative proof which avoids this, by using deeper properties of the semigroup  $S_C$ . Thus, writing  $2 - b(\alpha)$  instead  $\chi(\dot{E}_\alpha)$ , the above formula holds also in the non-complex case.

We conclude this paper giving an illustrative example.



**Example.** Let  $x, y$  be independent variables and set  $T = \mathbb{C}[x, y]_{(x, y)}$ . Consider the set  $V = \{\nu_1, \nu_2, \nu_3\}$  of divisorial valuations of  $\mathbb{C}(x, y)$  centered at  $T$ , whose minimal resolution is given by the sequence of ideals:

- $\nu_1$ :  $\mathfrak{m}_0 = \langle x, y \rangle, \mathfrak{m}_1 = \langle x, \frac{y}{x} \rangle, \mathfrak{m}_2 = \langle x, \frac{y}{x^2} - 1 \rangle$ ;
- $\nu_2$ :  $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_3 = \langle \frac{y}{x}, \frac{x^2}{y} \rangle, \mathfrak{m}_4 = \langle \frac{y}{x}, \frac{x^3 - y^2}{y^2} \rangle, \mathfrak{m}_5 = \langle \frac{x^3 - y^2}{y^2}, \frac{y^3}{x(x^3 - y^2)} \rangle, \mathfrak{m}_6 = \langle \frac{x^3 - y^2}{y^2}, \frac{y^5}{x(x^3 - y^2)^2} - 1 \rangle$ ;
- $\nu_3$ :  $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_3, \mathfrak{m}_4, \mathfrak{m}_5$ .

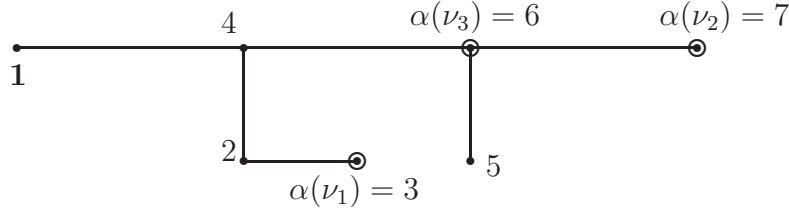


Figure 2: Dual graph

The dual graph of  $V$  has the shape of Figure 2. From it, we can deduce the values  $\underline{\nu}^1 = (1, 4, 4)$ ,  $\underline{\nu}^2 = (2, 6, 6)$ ,  $\underline{\nu}^3 = (3, 6, 6)$ ,  $\underline{\nu}^4 = (3, 12, 12)$ ,  $\underline{\nu}^5 = (3, 13, 13)$ ,  $\underline{\nu}^6 = (6, 26, 26)$ ,  $\underline{\nu}^7 = (6, 27, 26)$ , as well as the values  $\chi(E_\alpha) = 2 - b(\alpha)$  (Corollary 3), giving the following expression for the Poincaré series

$$P_V = \frac{(1 - t_1^3 t_2^{12} t_3^{12})(1 - t_1^6 t_2^{26} t_3^{26})}{(1 - t_1 t_2^4 t_3^4)(1 - t_1^3 t_2^6 t_3^6)(1 - t_1^3 t_2^{13} t_3^{13})(1 - t_1^6 t_2^{27} t_3^{26})}.$$

Moreover, by Theorem 5, the set  $\Lambda = \{Q_1 = x, Q_3 = y - x^2, Q_5 = y^2 - x^3, Q_7 = (y^2 - x^3)^2 - x^5 y\}$  is a minimal generating sequence of  $V$  since the set  $\{1, 3, 5, 7\}$  is the set of dead ends of the displayed dual graph.

## References

- [1] N. A'Campo, La fonction zêta d'une monodromie, *Comment. Math. Helv.* **50** (1975), 233-248.
- [2] A. Campillo, *Algebroid curves in positive characteristic*. Lect. Notes in Math. 813, Springer-Verlag, 1980.
- [3] A. Campillo, F. Delgado, S.M. Gusein-Zade, The extended semigroup of a plane curve singularity, *Proc. Steklov Inst. Math.* **221** (1998), 139-156.
- [4] A. Campillo, F. Delgado, S.M. Gusein-Zade, On generators of the semigroup of a plane curve singularity, *J. London Math. Soc.* **60** (2) (1999), 420-430.

- [5] A. Campillo, F. Delgado, S.M. Gusein-Zade, On the monodromy of a plane curve singularity and the Poincaré series of its ring of functions, *Funct. Anal. Appl.* **33** (1) (1999), 56-57.
- [6] A. Campillo, F. Delgado, S.M. Gusein-Zade, The Alexander polynomial of a plane curve singularity via the ring of functions on it, *Duke Math. J.* **117** (2003), 125-156.
- [7] A. Campillo, C. Galindo, The Poincaré series associated with finitely many monomial valuations, *Math. Proc. Cambridge Phil. Soc.* **134** (3) (2003), 433-443.
- [8] F. Delgado, The semigroup of values of a curve singularity with several branches, *Manuscripta Math.* **59** (1987), 347-374.
- [9] F. Delgado, *An arithmetical factorization for the critical point set of some maps from  $\mathbb{C}^2$  to  $\mathbb{C}^2$* . In: Singularities–Lille 1991. Jean-Paul Brasselet Ed. London Math. Soc. Lecture Note Series, 201. Cambridge University Press, 61–100, 1994.
- [10] F. Delgado, C. Galindo, A. Nuñez, Saturation for valuations on two-dimensional regular local rings, *Math. Z.* **234** (2000), 519- 550.
- [11] F. Delgado, S. M. Gusein-Zade, Poincaré series for several divisorial valuations, *Proc. Edinb. Math. Soc.* **46** (2) (2003), 501-509.
- [12] F. Delgado, H. Maugendre, Special fibres and critical locus for a pencil of plane curve singularities. *Compositio Math.* **136** (1) (2003), 69-87.
- [13] D. Eisenbud and W. Neumann, Three-dimensional link theory and invariants of plane curve singularities. Ann. of Math. Studies 110, Princeton Univ. Press, Princeton, NJ, 1985.
- [14] C. Galindo, On the Poincaré series for a plane divisorial valuation, *Bull. Belg. Math. Soc.* **2** (1995), 65-74.
- [15] S. Greco and K. Kiyek, *General elements of complete ideals and valuations centered at two dimensional regular local rings*, Algebra, Arithmetics and Geometry with Applications, Springer, 381-455, 2003.
- [16] R. Lazarsfeld, *Positivity in Algebraic Geometry, Vol. II*. Springer, 2004.
- [17] J. Lipman and K. Watanabe, Integrally closed ideals in two-dimensional regular local rings are multiplier ideals, *Math. Res. Lett.* **10** (2003), 1-12.
- [18] M. Spivakovsky, Valuations in function fields of surfaces, *Amer. J. Math.* **112** (1990), 107-156.
- [19] B. Teissier, Valuations, deformations, and toric geometry, Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999), *Fields Inst. Commun.* **33** (2003), 361-459. Amer. Math. Soc., Providence, RI.
- [20] O. Zariski, *Le problème des modules pour les branches planes, with an appendix by B. Teissier*. Hermann, Paris, 1986.